Feshbach Resonances in Rotating Fermi ~~(and Bose)~~ Condensates

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- Numerical work, in collaboration with E. H. Rezayi, CSULA.

Atoms in trap:

$$H^0 = \hbar\omega_0 \Big(\sum_i \underbrace{\sum_{m=0,\pm1}}_{} (n_{im} + 1/2) \Big)$$

particle label

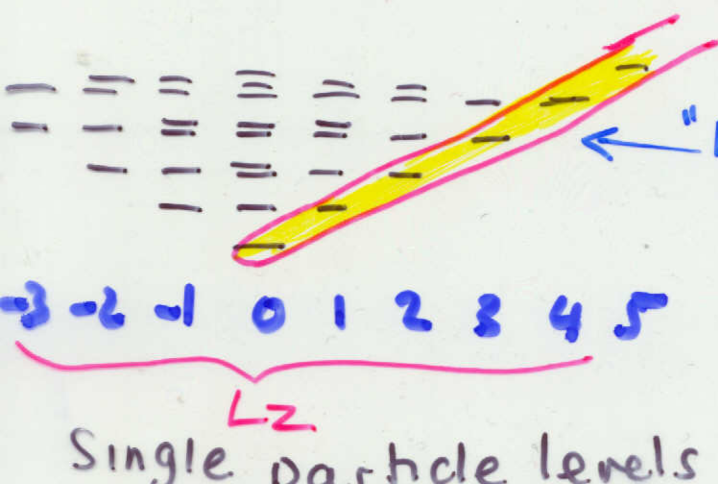
azimuthal angular momentum of oscillator mode

$$\hbar L^z = \hbar \sum_{i=1}^{N} \sum_{m=0,\pm1} (n_{im} + 1/2)$$

$$\boxed{H^0 \geq \omega_0 \left(|L^z| + 3/2 \right)}$$

analogs of "Lowest Landau level" states:

$$\boxed{\begin{array}{l} L^z \geq 0 \\ H^0 = \hbar\omega_0 (L^z + 3/2) \end{array}}$$



Single particle levels

"Lowest Landau level states"

$z = x + iy$

$$\boxed{\begin{array}{l} \psi_m = z^m e^{-1/4 |z|^2/\ell^2} \\ m = 0, 1, 2 \ldots \end{array}}$$

$\ell = \sqrt{\hbar / m\omega_0}$ oscillator length

- for a given azimuthal angular momentum L^z, "Lowest Landau level" (analog) states minimise H.

- only the $\ell^z = +1$ mode $b^\dagger_{+i}$ is excited

$$|\psi\rangle = \prod_{i=1}^{N} \frac{(b^\dagger_{+i})^{n_i}}{\sqrt{n_i!}} |vac\rangle$$

$$E = \hbar\omega \sum_{i=1}^{N} (n_i + 1/2)$$

$0 \leq n_i \leq N_\Phi \leftarrow$ Largest occupied L^z angular momentum single particle state.

uniform density dist $\longrightarrow$

$$L^z_{TOT} = \frac{1}{2} N N_\Phi$$

$$q N_\Phi = p N + r$$

$$\nu = p/q$$

"quantum Hall" Incompressible Fluid states occur when

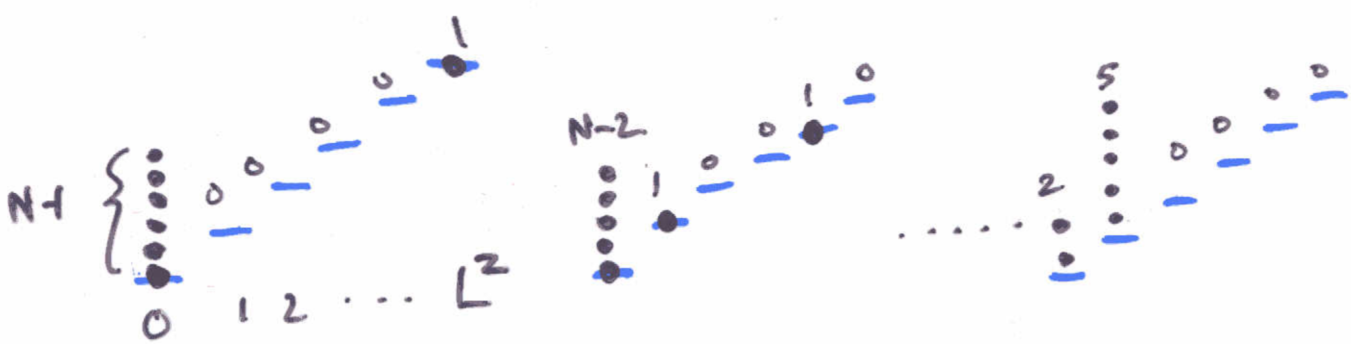
$$L^z \sim \frac{\nu}{2} N^2$$

$$\underbrace{2L^z/N}_{\text{number of vortices}} \sim \nu/2$$

(rational) "Filling factor"

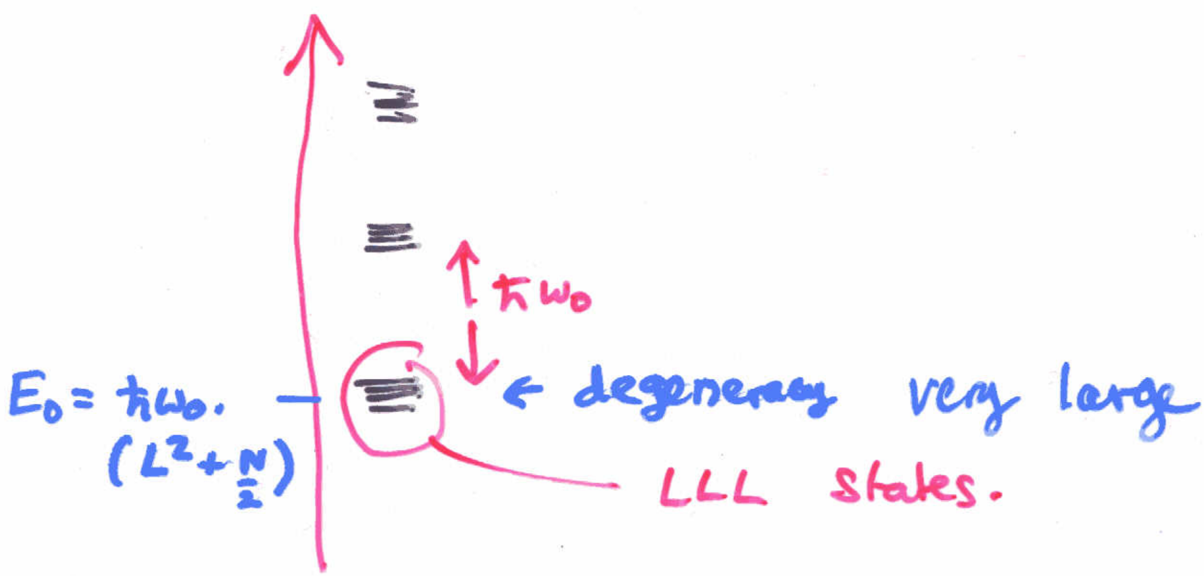
$$\nu = \frac{\text{particles}}{\text{vortices}}$$

Lowest Landau Level states of Bosons with total angular momentum L^z_{TOT}



$L^z = 5, \; N = 7$

Number of configurations with given L^z, N (Spinless bosons)



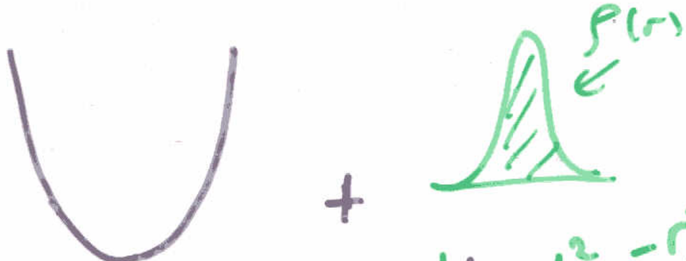
- If interactions are "weak" compared to $\hbar\omega_c$, can work within Lowest Landau level sector only.

– what does "weak" interaction mean?

– $\boxed{\text{not strong enough to add anharmonic terms to harmonic well potential}}$

- **Gross - Pitaevski:** $H_{int} = g\int d^3r\, \psi^{\dagger 2}(r)\, \psi(r)$

$\uparrow$ dimension energy $\times$ volume



$\underbrace{\tfrac{1}{2} m \omega_0^2 r^2}_{\tfrac{1}{2}\tfrac{\hbar^2}{m\ell^2}(r/\ell)^2} \qquad N g |\psi(0)|^2 e^{-r^2/4\ell^2} \qquad |\psi(0)|^2 = \left(\frac{1}{2\pi\ell^2}\right)^{3/2}$

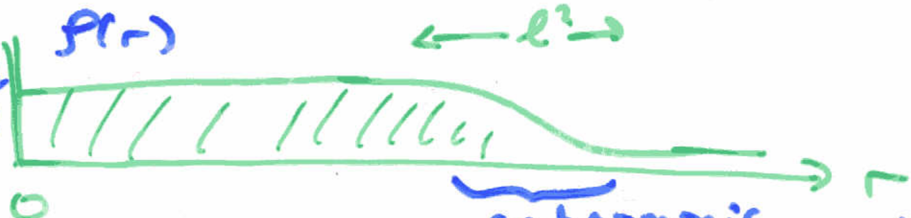
"weak coupling" of the non-rotating ideal bose gas means

$$\frac{1}{N}\,\frac{\hbar^2}{m\ell^2} \gg g/\ell^3$$

very small range of validity

- "quantum Hall incompressible state"

Not $O(N)$ $\rightarrow \left(\frac{\nu}{2\pi\ell^2}\right)^{3/2}$



anharmonic corrections to well only occur at edge...

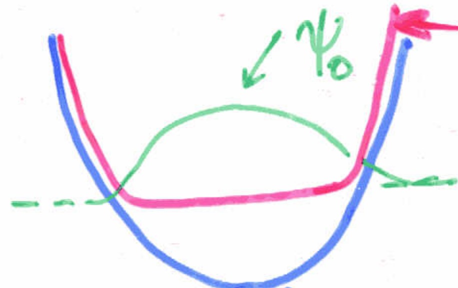
$g = V_0 a^3$

$$\underbrace{\hbar\omega_0 = \frac{\hbar^2}{m\ell^2} \gg g/\ell^3}_{\text{Not } O(1/N)}$$

$$\boxed{\hbar\omega_0 \gg V_0 (a/\ell)^3}$$

Spin the condensate (bosons)

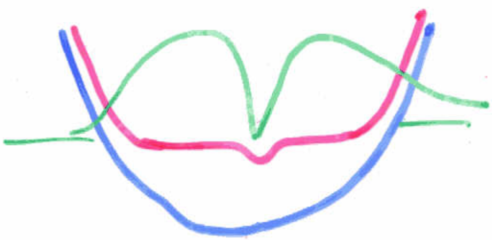
(a) $L^2 = 0$ Gross-Pitaevsky Limit



renormalized (mean field) potential (non-linear Schrödinger eqn.)

bare potential (harmonic)

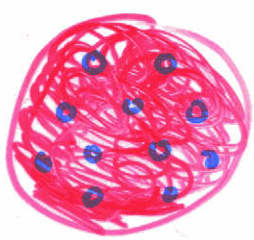
(b) one vortex GP solution:



$L^2 = N$ (vortex at center of trap)

top view

(c) $N_\Phi \sim O(1)$ vortices, lattice



vortices repel each other, form lattice

$L^2 = \frac{1}{2} N N_\Phi$

because vortices are uniformly spread out, not all at center (analog of type II superconductors) (d): The bose system minimizes its energy by being in the lowest landau level (assumes interaction potential is weak compared to $\hbar\omega_0$)

(e) quantum uncertainty of position of 2D vortices:

$$[R^x, R^y] = i a^2$$

$2\pi a^2 \equiv$ area per particle of the underlying quantum fluid

(FDMH, YS-Wu 1985)

- quantum "melting" of the lattice by zero-point fluctuations occurs when amplitude of zero-point motion is comparable to lattice spacing

or $\dfrac{N_\Phi}{N} \gtrsim \dfrac{1}{6}$

numerical study by Cooper, Wilkin + Gunn Phys Rev Lett 87, 120405 (2001)

Note!

Vortices → charged particles
particle density of fluid → magnetic flux:

Same criterion $\dfrac{N_{particles}}{N_{flux}}$ works in low density Landau level!

Principal

- λ quantum fluid states appear to be similar to "Read-Rezayi parafermion states"

$$\nu = \frac{P}{2} : \left(\tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \cdots 6 \right)$$

$$\Psi = \mathrm{Sym}\left\{ \prod_{\alpha=1}^{k} \prod_{i<j} (z_{i\alpha} - z_{j\alpha})^2 \right\} \prod_i e^{-\frac{1}{4} r^2/\ell^2}$$

$z = x+iy$ $\qquad$ $i,j = 1 \ldots N/k$

oscillator ground state $|\Psi_0\rangle$

- k copies of $\nu = 1/2$ Laughlin state, then symmetrized.

$k=1 \longrightarrow$ Laughlin state of bosons

$k=2 \longrightarrow$ Moore-Read paired state.

- For "contact" pseudopotentials (s-wave scattering only)

The $\nu = 1/2$ Laughlin state is the lowest density quantum Hall state (of bosons)

$$\prod_{i<j} (z_i - z_j)^2 \cdot \Psi_0$$

NOTE! Quantum Hall condensates are interpreted as analogs of bose condensates via Chern-Simons Theory. (Note: at $\nu = \frac{1}{\text{odd}}$ This is "bosonization of fermions" ....)

$$\Psi^{(m)}_{\text{Laughlin}}\left(\begin{smallmatrix}N+1\\ \text{particles}\end{smallmatrix}\right) = \left(\prod_{j=1}^{N}(z_{N+1}-z_j)\right)^m \Psi_0(z_n)\ \Psi^{(m)}_N$$

adds m vortices as well as the particle

This is The usual term in BEC (add an extra particle to the GS

"composite bosons"

The condensate is of

$\boxed{\text{particle + m-vortex composites}}$

NOT the BARE particles

- even if The bare particle is a Fermion (m=odd) The composite is a "boson"!
- The vortices cancel The rotation of the particle: The composite has "plane wave" (straight-line motion states → it can condense!

Feshbach resonance

$H =$ (free spin-$\frac{1}{2}$ fermions)
$+$ (free spinless bosons)
$+$ Feshbach process

$$\psi^{\dagger}_{\uparrow}(x)\,\psi^{\dagger}_{\downarrow}(x') \longrightarrow \phi^{\dagger}(x)$$

pair of fermions mass M — bosons. mass $2M$

$\boxed{\text{$\nu = \frac{1}{2}$ state of bosons (Laughlin) has same density as $\nu = 2$ (filled Landau level) of spin-$\frac{1}{2}$ fermions}}$

$\nu = \frac{1}{2}$ Laughlin state of bosonic pairs $\quad \xleftrightarrow{\text{Feshbach}} \quad$ $\nu = 2$ filled Landau level of spin-$\frac{1}{2}$ fermions

$\Delta < 0$ $\qquad\qquad$ $\Delta > 0$

QHE analog of Bose condensate

"composite boson condensate"

slater ~~determin~~ determinant (Like BCS state)

- making the $\nu = 2$ wavefunction for Fermions "look like" $\nu = 1/2$ bosonic pairs

① $$\Psi = \sum_{\sigma_i = \pm 1/2} \prod_{i<j} (z_i - z_j)^{\delta_{\sigma_i \sigma_j}} \underbrace{e^{i\frac{\pi}{2}\,\mathrm{sgn}(\sigma_i - \sigma_j)}}_{\text{antisymmetric}} \prod_i u_i^{\frac{1}{2}+\sigma_i} v_i^{\frac{1}{2}-\sigma_i}$$

$\nu = 2$

$i = 1 \ldots N$

$z_i = x_i + i y_i$ complex coordinate in xy plane

$u_i = \uparrow$, spinor "coordinate" for atomic spin

$v_i = \downarrow$

$i = 1 \ldots N/2$

② $$\Psi = A\left\{ \prod_{i<j} (z_{2i-1} - z_{2j-1})(z_{2i} - z_{2j}) \prod_i \overset{\uparrow}{u_{2i-1}} \overset{\downarrow}{v_{2i}} \right\}$$

antisymmetrizer

③ $$\Psi = A\left\{ \prod_{i<j} (\underbrace{z_{2i-1} + z_{2i}}_{\substack{\text{center of mass}\\ \text{of pair: } 2i-1,\,2i}} - \underbrace{z_{2j-1} - z_{2j}}_{\substack{\text{center of mass}\\ \text{of pair: } 2j-1,\,2j}})^2 \prod_i \underbrace{(\overset{\uparrow\downarrow}{u_{2i-1} v_{2i}} - \overset{\downarrow\uparrow}{v_{2i-1} u_{2i}})}_{\text{explicitly spin singlet}}\right.$$

$i = 1 \ldots N/2$

$\Downarrow$ Feshbach

④ $$\Psi = \prod_{i<j} (z_i - z_j)^2$$

(molecules)

Laughlin

$\prod_{i<j} (z_i - z_j) = \det_{i,j}(z_i^{j-1})$ vandermonde

like rewriting BCS wavefunction in various ways…

Feshbach term

$$H = \underbrace{\sum_i \left(\frac{P_i^2}{2m} + \frac{1}{2} m\omega_0^2 r_i^2\right)}_{\substack{\text{atoms}\\ \text{(fermions)}\\ \text{mass } m}} + \underbrace{\sum_i \frac{P_i^2}{4M} + \frac{1}{2}\cdot 2M\omega_0^2 R_i^2}_{\substack{\text{molecules,}\\ \text{(bosons,}\\ \text{mass } 2M)}} \; {+\Delta_0}$$

- same well frequency, Kohn mode (CM) decouples from relative motion in well

- $$H^{fesh} = \int d^3x \int d^3x' \, g(x,x') \underbrace{\Phi^+\!\left(\tfrac{x+x'}{2}\right) \psi_\uparrow(x) \psi_\downarrow(x')}_{\text{conserves center of mass}} + h.c.$$

We MUST include explicit cutoff, $g(x,x') = g\delta^3(x-x')$ is not valid: (for $\Delta_0 < 0$)

- $g(x,x') \sim$ internal molecular wavefunction of the "Feshbach boson"....

- In the absence of the well, Feshbach shifts involve

$$\int d^3x \int d^3x' \, |g(x-x')|^2 \frac{e^{-K|x-x'|}}{|x-x'|} = \infty \text{ for } g(x) = g\,\delta^3(x)$$

- atom wavefunctions (LLL)

$$\psi(\vec{r}) = z^m e^{-\frac{1}{4} r^2/\ell^2}$$

- molecule LLL wavefunctions

$$\phi(r) = z^m e^{-\frac{1}{2} r^2/\ell^2}$$

not 1/4

$$H^{fesh} \cdot \phi(r) \longrightarrow g(r_1, r_2) \left(\frac{z_1+z_2}{2}\right)^m e^{-\frac{1}{2}\left|\frac{r_1+r_2}{2}\right|^2/\ell^2}$$

$r = r_1 + r_2$

$$= g(r_1, r_2)\, e^{+\frac{1}{8}\frac{|r_1-r_2|^2}{\ell^2}} \underbrace{\left(\frac{z_1+z_2}{2}\right)^m e^{-\frac{1}{4}\frac{r_1^2+r_2^2}{\ell^2}}}$$

a LLL atomic pair wavefunction, relative angular momentum 0.

FESHBACH WITH

$$g(r_1, r_2) = g\, e^{-\frac{1}{8}\frac{|r_1-r_2|^2}{\ell^2}}$$

Conserves The LLL property

- This work is inspired by N.R. Cooper, Cond-mat preprint (2003) (discussion of Feshbach and rotating bosonic atoms + molecules)

(— Cooper uses $g(r_1, r_2) = g\,\delta^3(r - r')$, which leads To inconsistencies in what is otherwise a very interesting paper)

LLL - conserving Feshbach term

$$H^{fesh} = V_{fesh} \int d^3x \int d^3x' \left(\frac{e^{-\frac{1}{8}\frac{|x-x'|^2}{\ell^2}}}{(\pi \ell^2)^{3/4}} \right) \Phi^\dagger\left(\tfrac{x+x'}{2}\right) \psi_\uparrow(x)\,\psi_\downarrow(x') + h.c.$$

dimensionally an energy

$$H^{fesh} = \sum_{\substack{\text{pair states}\\ (\text{relative angular momentum})\\ 0}} V_{fesh}\, |\text{molecule}\rangle\langle\text{pair}| + h.c.$$

Same center of mass state

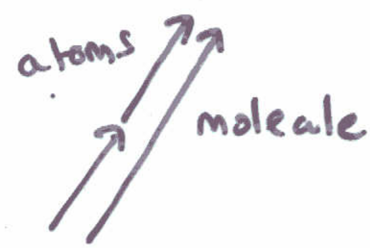
NOTE: THE ATOMIC CHARGE DENSITY IS NOT CHANGED BY THIS PROCESS!

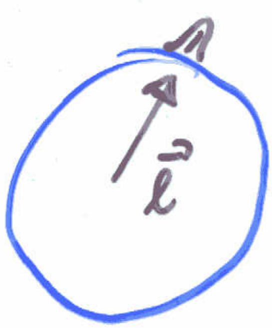
on sphere

atoms have angular momentum

$$\ell_{atom} = \frac{1}{2} N_\Phi$$

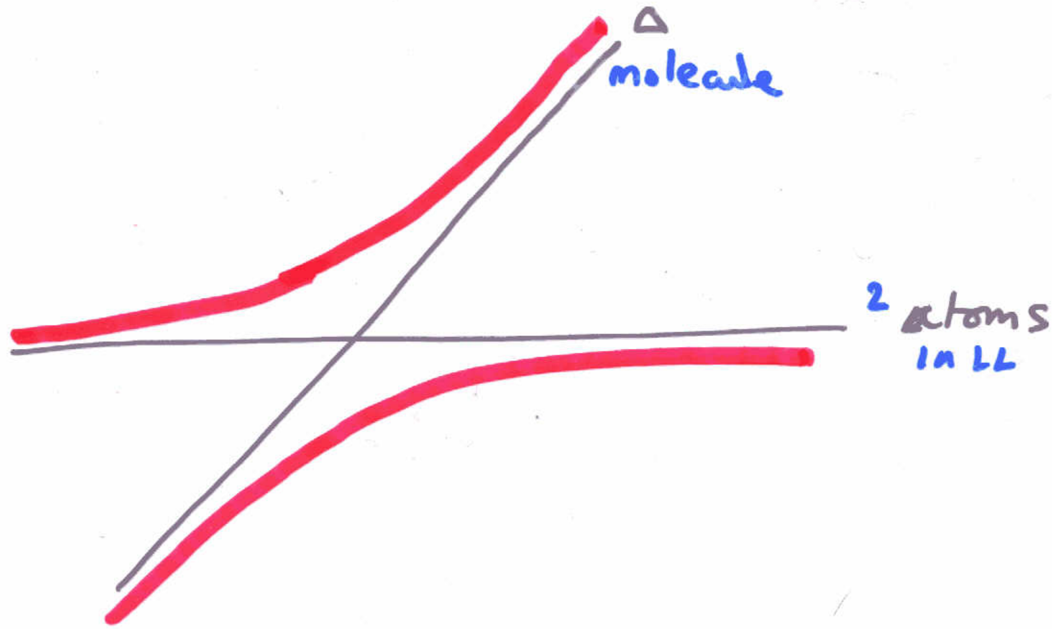
$$\ell_{molecule} = N_\Phi$$





"angular momentum" points to location of particle on the 2D surface of the sphere.

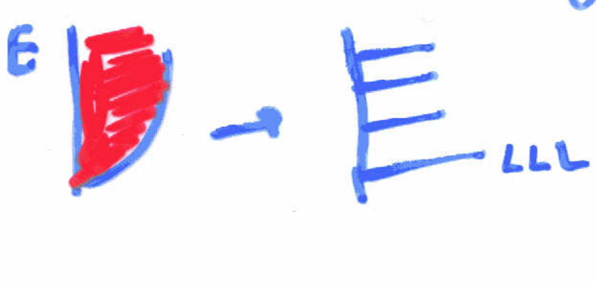
two atoms $\rightleftharpoons$ 1 molecule



here the 2-atom problem is just

$$H = \begin{bmatrix} 0 & V \\ V & \Delta \end{bmatrix} \begin{array}{l} |\text{atoms}\rangle \\ |\text{molecule}\rangle \end{array}$$

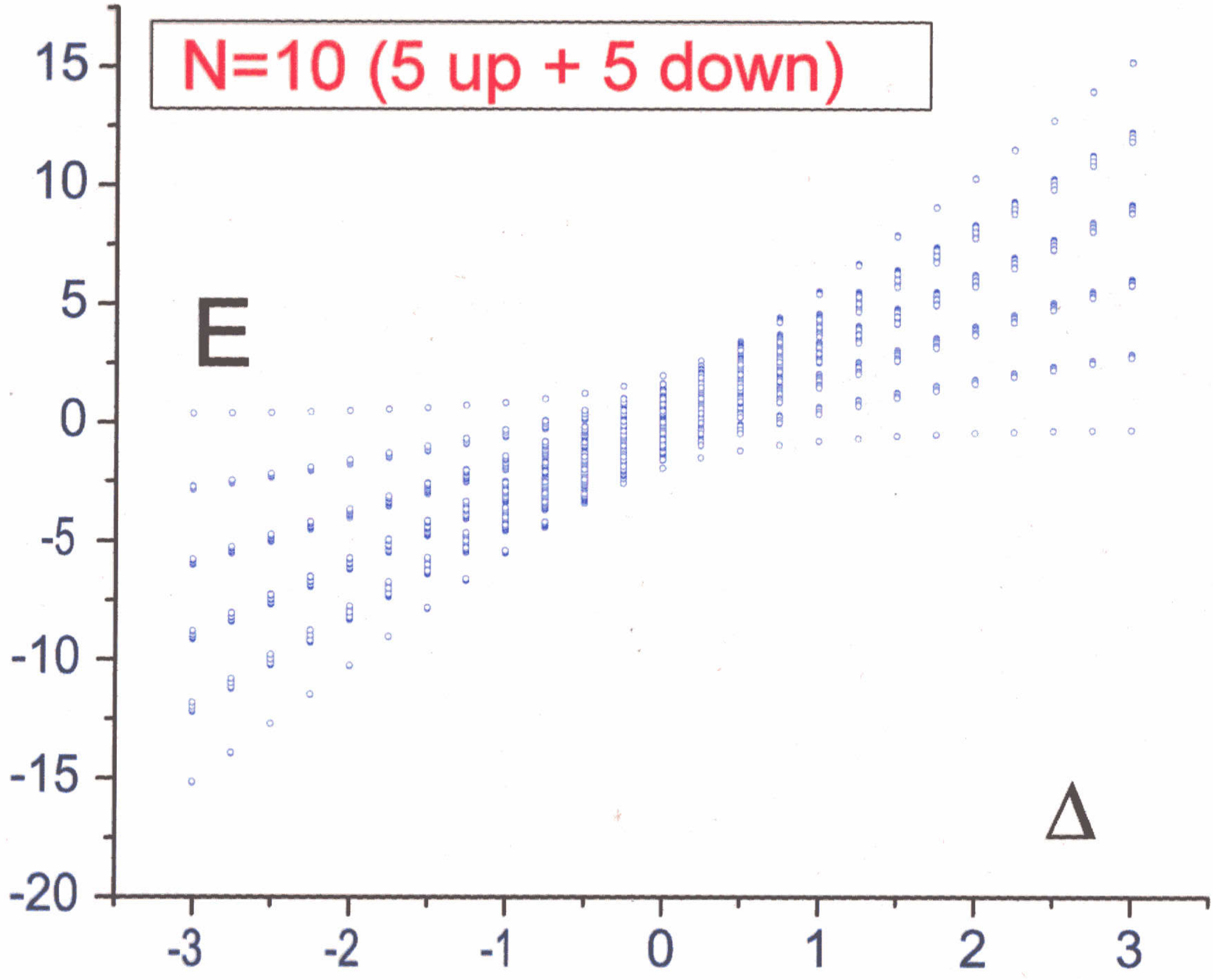
- The free fermi gas of the BEC/BCS problem has collapsed to a single Landau level.

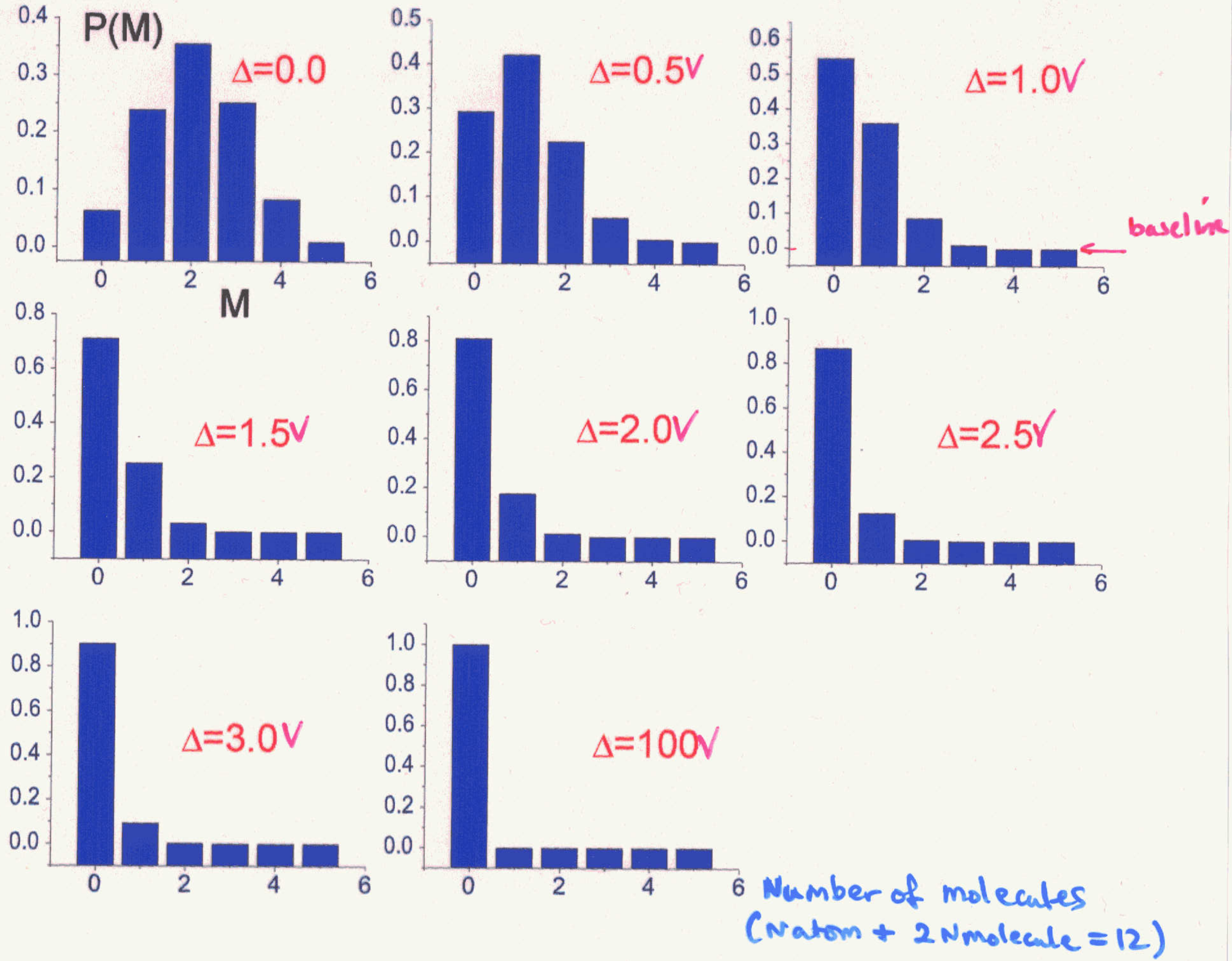


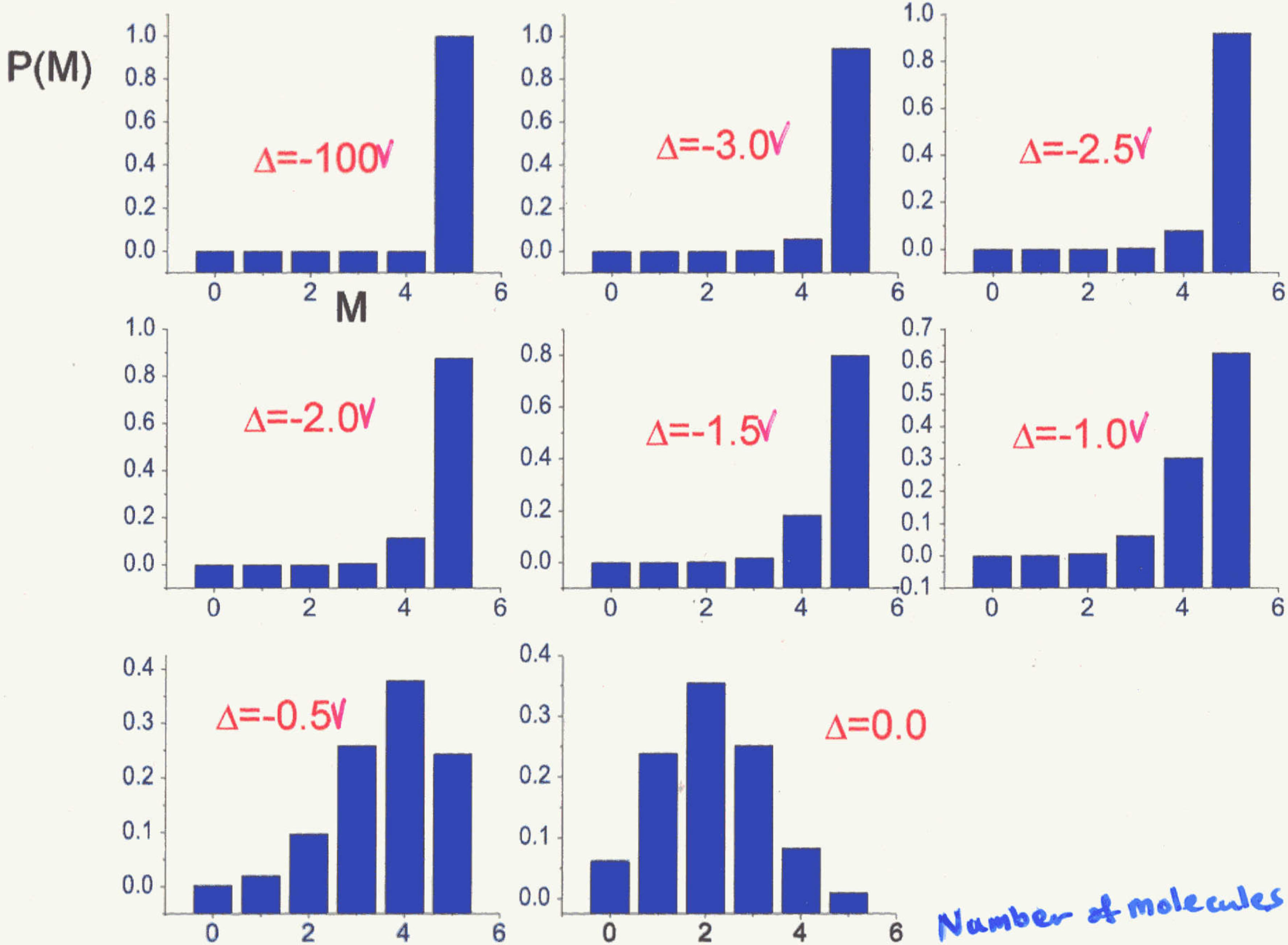
— no motion!

same ~~charge~~ atomic density as a pair of atoms

— but NO Pauli exclusion . . . .

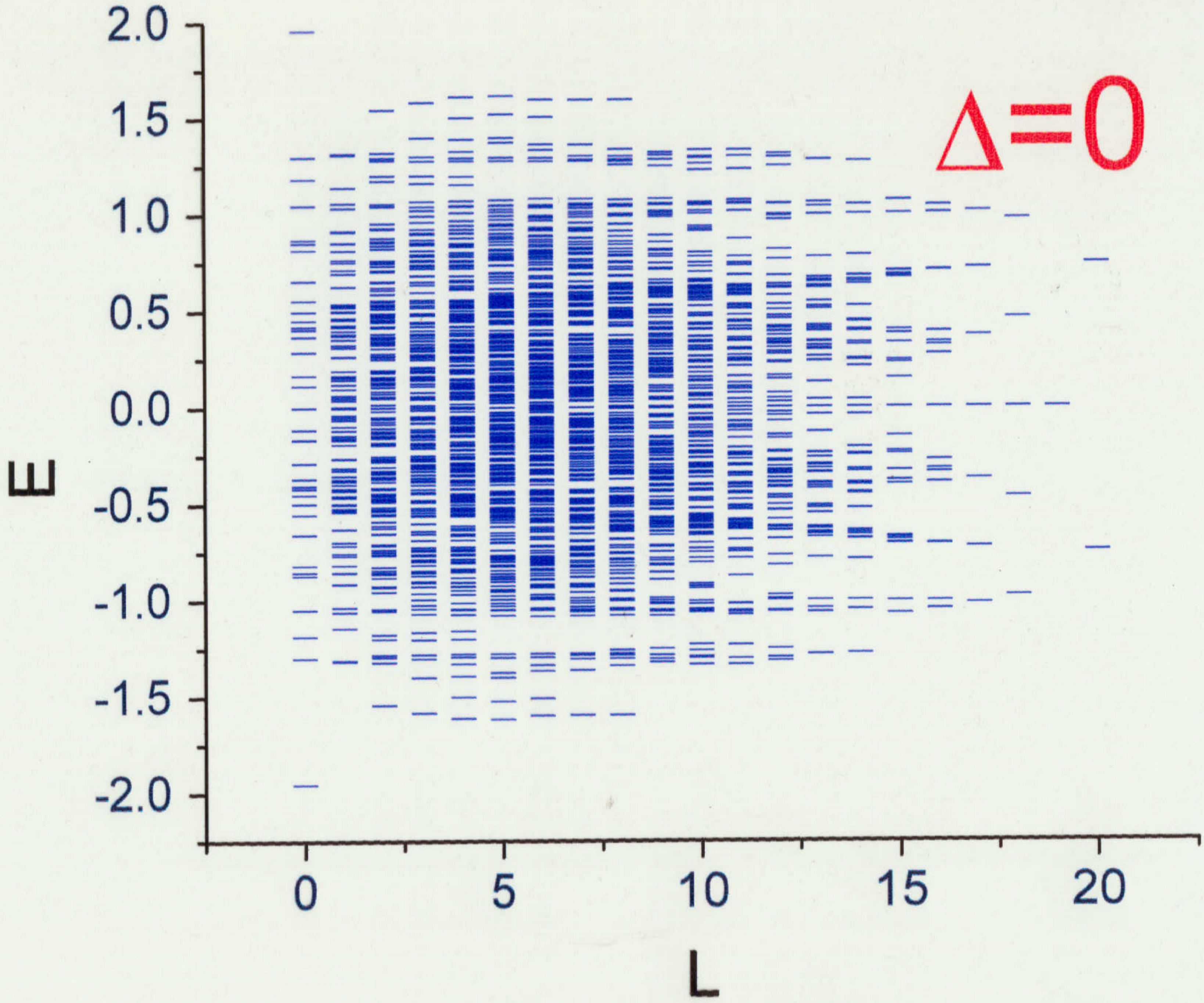

N=10 (5 up + 5 down)
E
Δ
15
10
5
0
-5
-10
-15
-20
-3
-2
-1
0
1
2
3

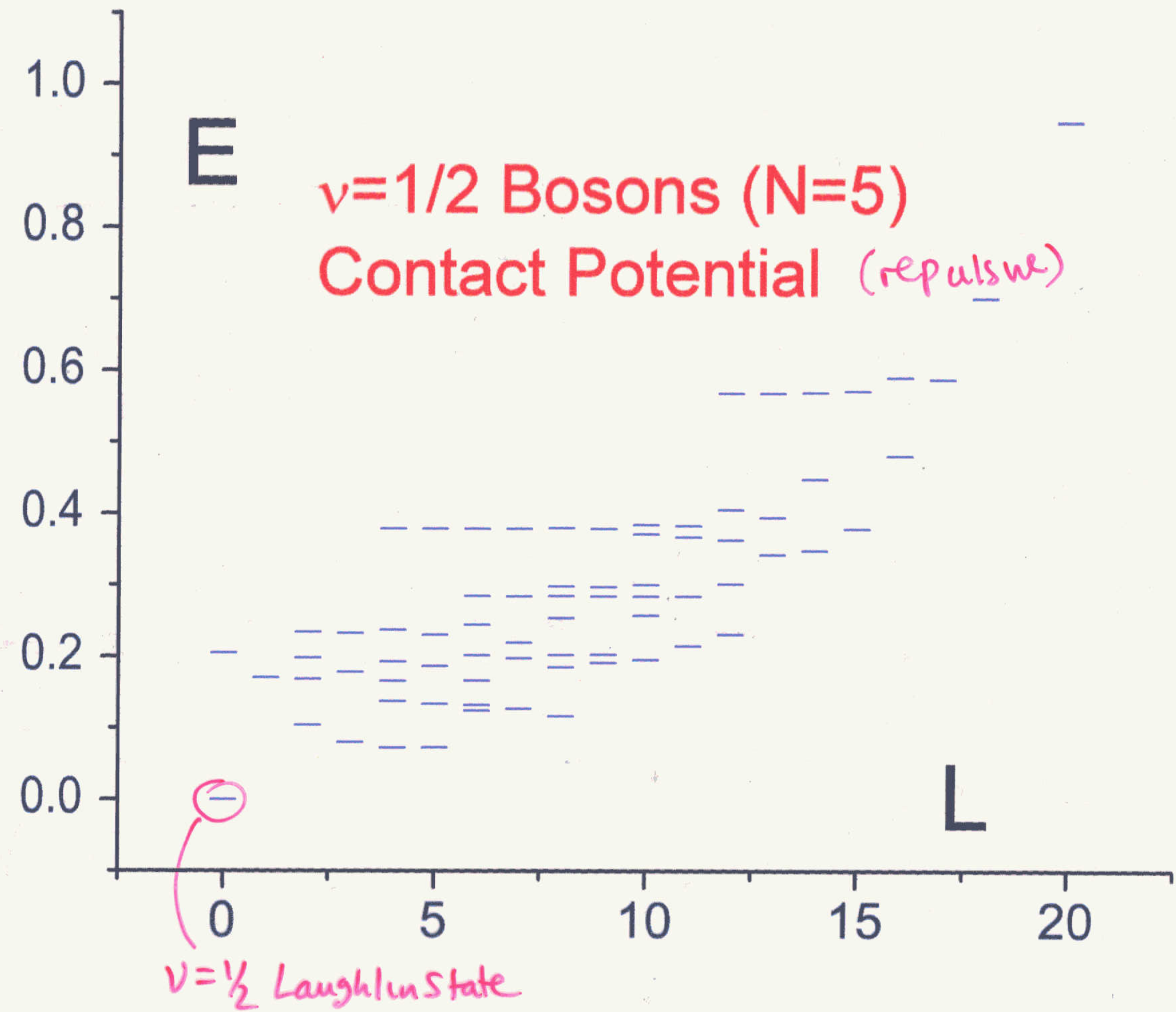

P(M)
M
Δ=0.0
Δ=0.5V
Δ=1.0V
baseline
Δ=1.5V
Δ=2.0V
Δ=2.5V
Δ=3.0V
Δ=100V
Number of molecules
(Natom + 2Nmolecule = 12)



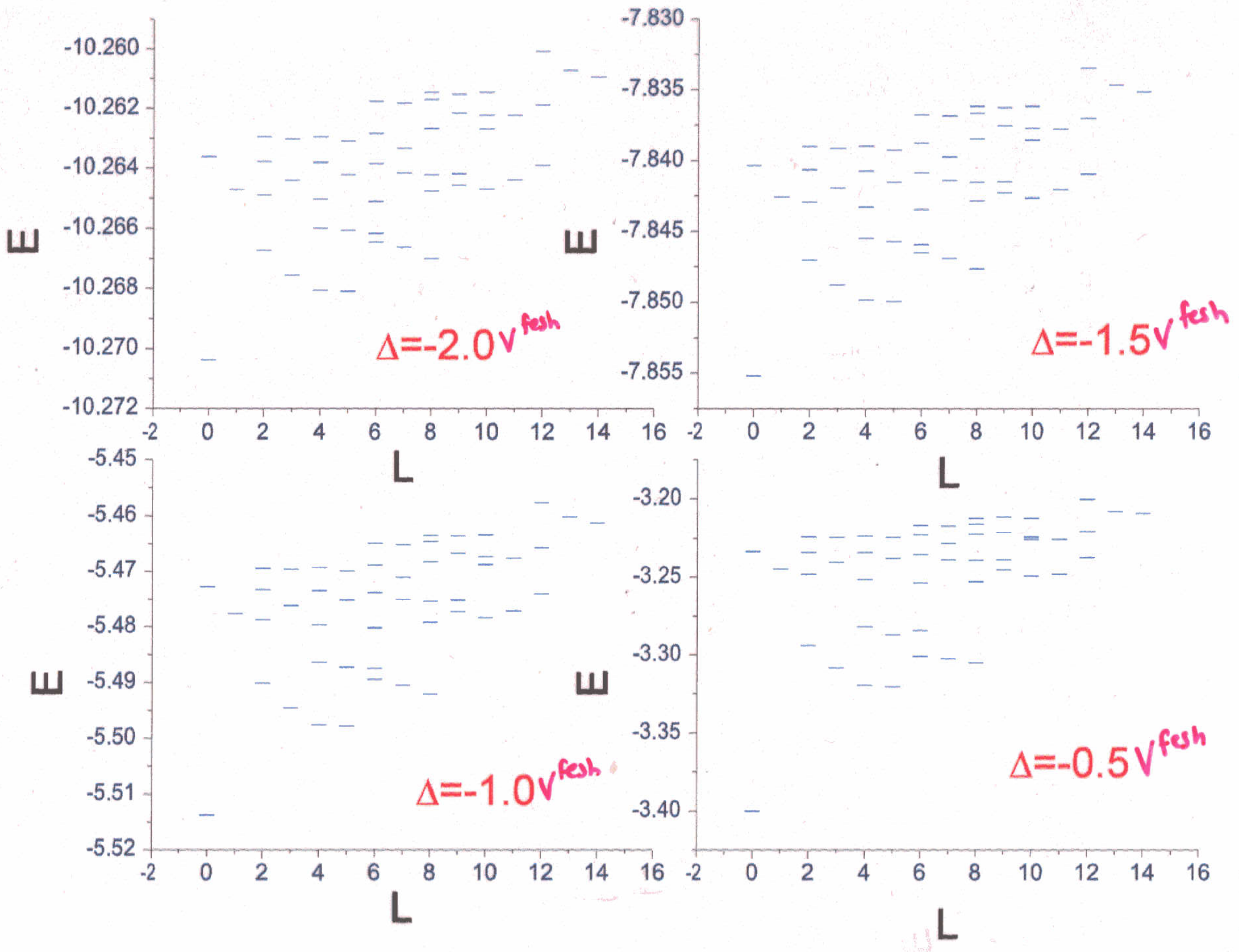
Number of molecules

$N_{atom} + 2N_{molecule} = 12$

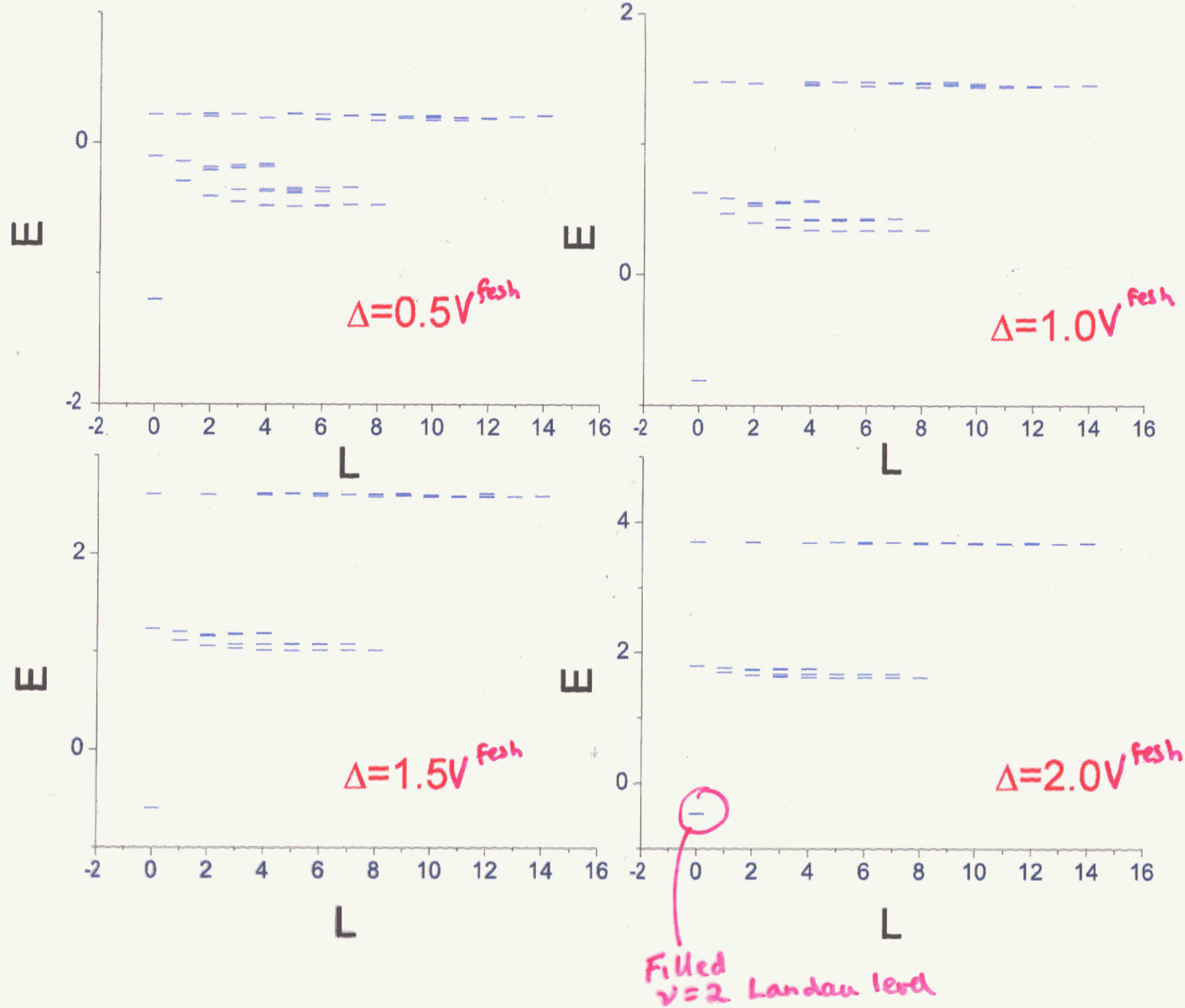

Δ=0
E
L
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
0
5
10
15
20



E
ν=1/2 Bosons (N=5)
Contact Potential
(repulsive)
1.0
0.8
0.6
0.4
0.2
0.0
0
5
10
15
20
L
ν=½ Laughlin state



E
L
Δ=-2.0V^fesh
Δ=-1.5V^fesh
Δ=-1.0V^fesh
Δ=-0.5V^fesh



E
L
$\Delta=0.5V^{fesh}$
$\Delta=1.0V^{fesh}$
$\Delta=1.5V^{fesh}$
$\Delta=2.0V^{fesh}$
Filled
$\nu=2$ Landau level

Projection of pure boson component of the ground-state wavefunction on the nu = 1/2 boson Laughlin wavefunction (5 particles)

Delta	Projection	pure boson fraction of WF
-3.00000	0.99904	0.9418555320222478
-2.75000	0.99905	0.9314871352360486
-2.50000	0.99906	0.9181590253996671
-2.25000	0.99908	0.9006697045111115
-2.00000	0.99910	0.8771700982856837
-1.75000	0.99913	0.8447242037222000
-1.50000	0.99918	0.7985177812616734
-1.25000	0.99927	0.7304308697909475
-1.00000	0.99941	0.6266876283211065
-0.75000	0.99961	0.4664569968240566
-0.50000	0.99985	0.2443179617552539
-0.25000	0.99997	0.0646425401922779
0.00000	1.00000	0.0096859991444091
0.25000	1.00000	0.0012311335167883
0.50000	1.00000	0.0001638627666901
0.75000	1.00000	0.0000250145720765
1.00000	1.00000	0.0000045235457075
1.25000	1.00000	0.0000009692243863
1.50000	1.00000	0.0000002425256186
1.75000	1.00000	0.0000000695398036
2.00000	1.00000	0.0000000224234377
2.25000	1.00000	0.0000000079963928
2.50000	1.00000	0.0000000031089457
2.75000	1.00000	0.0000000013021772
3.00000	1.00000	0.0000000005817591

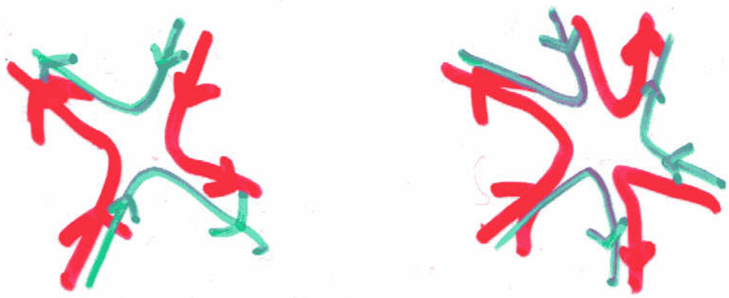
↑ all molecules

↓ all atoms

Conclusions (Feshbach)

- The "Feshbach boson" is a somewhat strange term. In our case, the process essentially is switching on and off the Pauli exclusion effect of the "closed channel" pair.

- We see the induced interactions between the molecules, not just two body, but three body corrections too, in the $\Delta \approx 0$ regime:



- Smooth crossover from $\nu = 2$ fermions to $\nu = 1/2$ boson Laughlin.

$\approx$ like BCS $\rightleftarrows$ BEC .....