Bosons and Fermions in optical lattices

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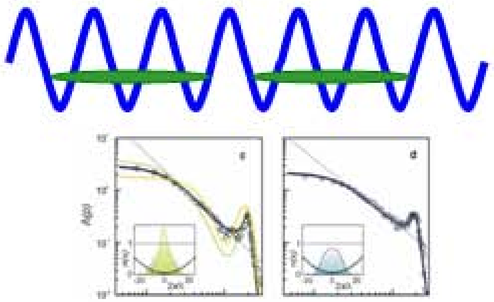
S. Fölling (Mainz)

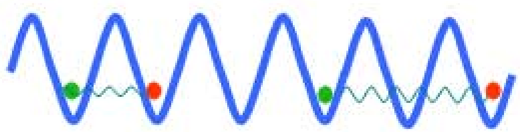
TRENTO, March 2004

OUTLINE

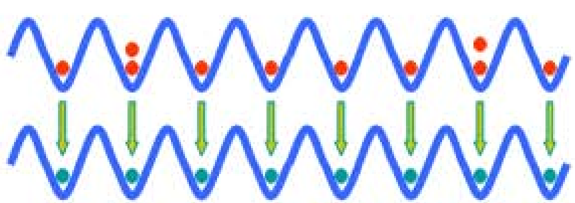
1. Tonks-Girardeau gas in 1D optical lattices:



2. Extensions:

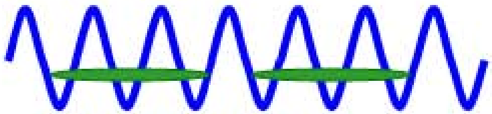


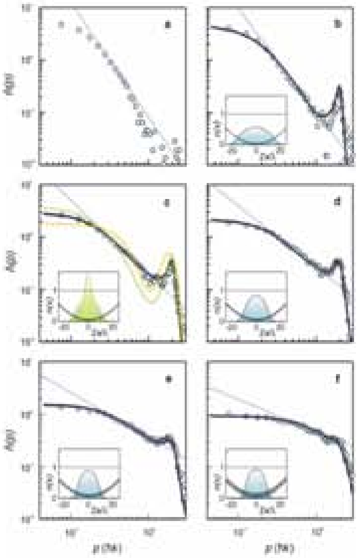
3. Loading optical lattices:



1. Tonks-Girardeau gas in OL

B. Paredes, A. Widera, V. Murg, O. Mandel, S. Fölling,
J.I. Cirac, G. V. Shlyapnikov, T. W. Hänsch & I. Bloch





M. Girardeau, J. Math. Phys. 1, 516 (1960).
Relationship between Systems of Impenetrable Bosons and Fermions in One Dimension.

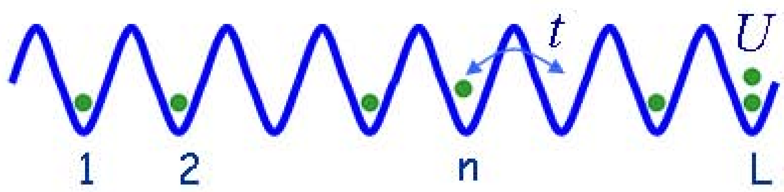
Other works in the field of Atomic Physics:

Experiments in the gas phase: NIST and Zürich

Theory in the gas phase: Olshanii et al., Cazalilla, Giorgini et al, Drummond et al, Shlyapnikov et al...
Theory in an optical lattice: Zwerger et al, ...

Fermionization: review

N particles, L sites



Characteristic energies:

t : tunnel

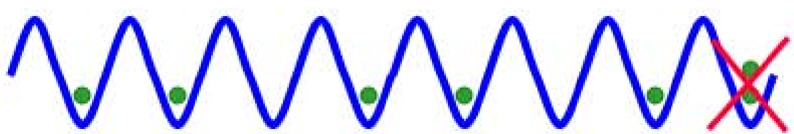
U : On-site interaction

Bose-Hubbard model: $H_B = -t\sum_n \left(a_{n+1}^\dagger a_n + a_n^\dagger a_{n+1}\right) + U\sum_n \left(a_n^\dagger\right)^2 a_n^2$

bosonic operators

- We will consider:
  - Strong interacting regime: $t \ll U$
  - Low temperature: $k_B T \ll U$
  - Dilute: $\nu = N/L < 1$

➡ At most one atom per site:

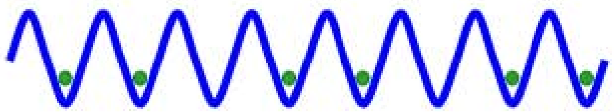


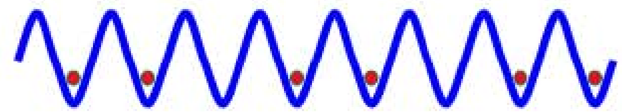
Hard-core bosons.

Fermionization

Hard-core Bosons | **Fermions**

Allowed states: | Allowed states:

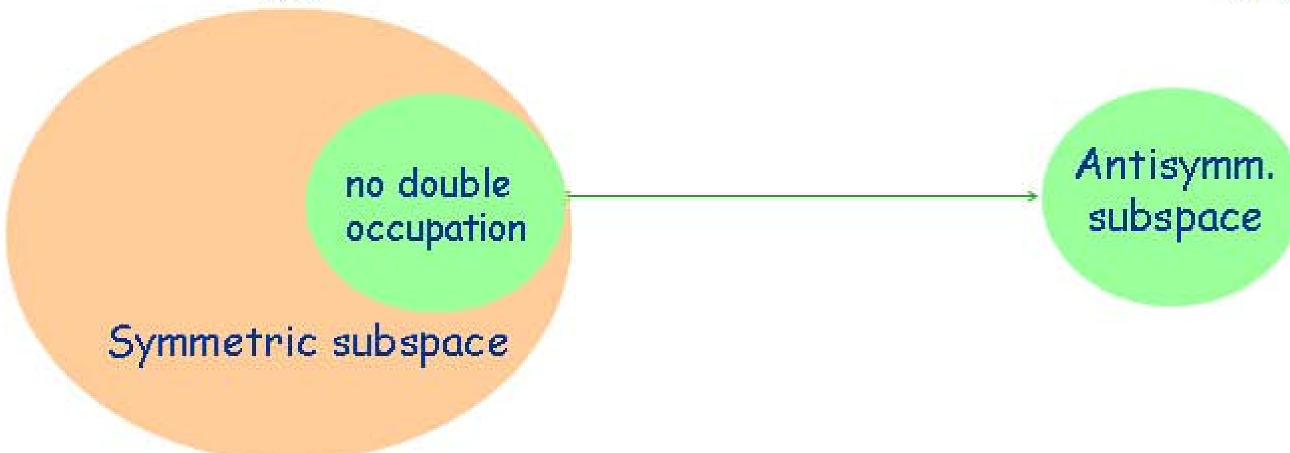




$$|1,1,0,1,1,0,1,1\rangle \longleftrightarrow |1,1,0,1,1,0,1,1\rangle$$

$$|1,0,1,1,1,0,1,1\rangle \longleftrightarrow |1,0,1,1,1,0,1,1\rangle$$

etc | etc



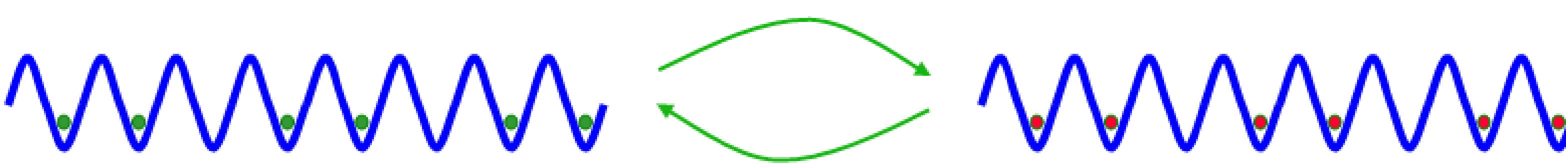
We can establish a one-to-one correspondence

... but we must be carefull, since states (H) have different symmetries

JORDAN-WIGNER TRANSFORMATION

Hard-core Bosons		Fermions
STATES	$\longleftrightarrow$	STATES
$\|1,1,0,1,1,0,1,1\rangle$		$\|1,1,0,1,1,0,1,1\rangle$
OPERATORS (OBSERVABLES)	$\longleftrightarrow$	OPERATORS (OBSERVABLES)
a_n		$(-1)^{\sum_{m<n} c_m^{\dagger} c_m} \, c_n$

It ensures that $\langle \Psi_B \,|\, O_B \,|\, \Psi_B \rangle = \langle \Psi_F \,|\, O_F \,|\, \Psi_F \rangle$



STATE IN THERMAL EQUILIBRIUM

$$a_n \longleftrightarrow (-1)^{\sum_{m<n} c_m^\dagger c_m} c_n$$

Density: $a_n^\dagger a_n \longleftrightarrow c_n^\dagger c_n$

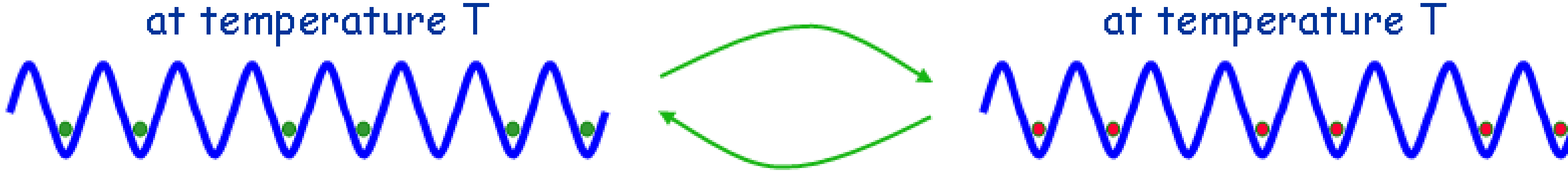
Tunnel: $a_{n+1}^\dagger a_n \longleftrightarrow c_{n+1}^\dagger (-1)^{c_n^\dagger c_n} c_n = c_{n+1}^\dagger c_n$

Hamiltonian: $$H_B = -t\sum_n \left(a_{n+1}^\dagger a_n + a_n^\dagger a_{n+1}\right) + U\sum_n \left(a_n^\dagger\right)^2 a_n^2 \longleftrightarrow H_F = -t\sum_n \left(c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1}\right)$$

Density operator: $\rho_B \propto e^{-\beta H_B} \longleftrightarrow \rho_F \propto e^{-\beta H_F}$

Strong interacting bosons
at temperature T

Free fermions
at temperature T



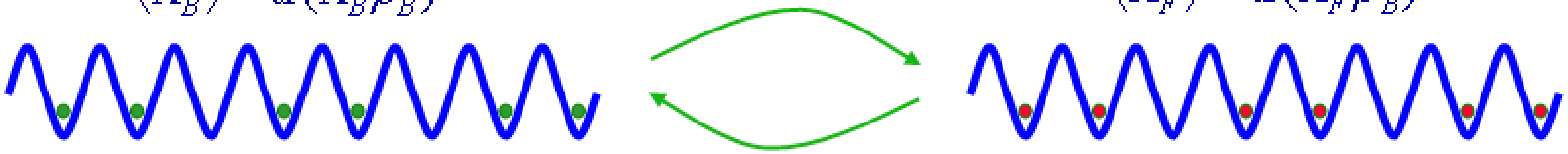
$\rho_B \propto e^{-\beta H_B}$

$\rho_F \propto e^{-\beta H_F}$

TONKS-GIRARDEAU GAS

OBSERVABLES

$\langle A_B \rangle = \mathrm{tr}(A_B \rho_B)$ $\qquad\qquad$ $\langle A_F \rangle = \mathrm{tr}(A_F \rho_B)$



A) The obsevable has the same expression:

Density: $a_n^\dagger a_n \longleftrightarrow c_n^\dagger c_n$

Tunnel: $a_{n+1}^\dagger a_n \longleftrightarrow c_{n+1}^\dagger c_n$

Hamiltonian: $H_B = -t \sum_n \left(a_{n+1}^\dagger a_n + a_n^\dagger a_{n+1} \right) \longleftrightarrow H_F = -t \sum_n \left(c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1} \right)$

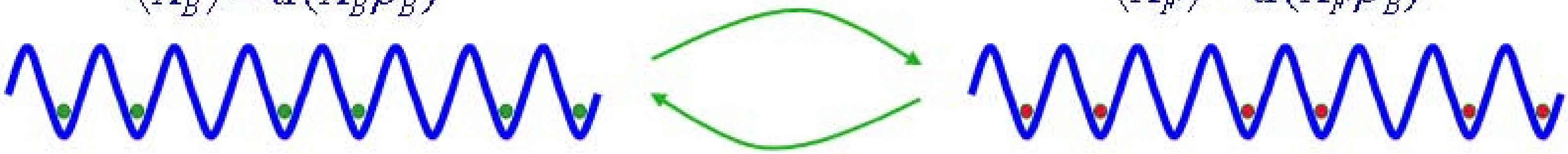
Examples:

Density and density fluctuations
Kinetic energy
Total energy and energy fluctuations
Spectrum
Density of states
Entropy
Etc.

For strong int. Bosons are temperature T
are the same as
for free Fermions at temperature T.

OBSERVABLES

$\langle A_B \rangle = \mathrm{tr}(A_B \rho_B)$ $\qquad\qquad$ $\langle A_F \rangle = \mathrm{tr}(A_F \rho_B)$



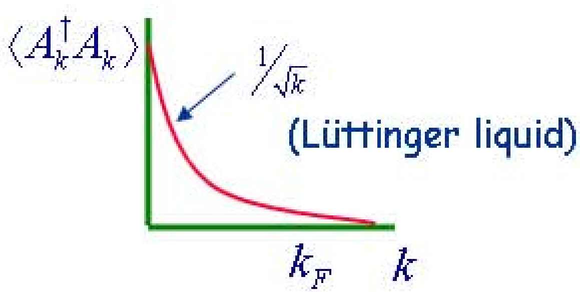
B) The obsevable has a different expression:

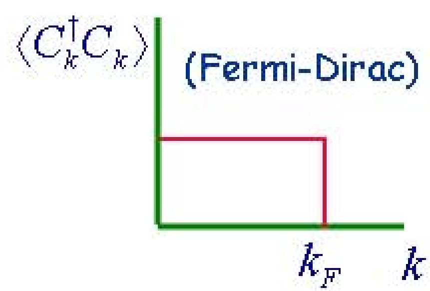
Correlation: $\quad a_m^\dagger a_n \longleftrightarrow c_m^\dagger (-1)^{\sum_{n<l<m} c_l^\dagger c_l} c_n \neq c_m^\dagger c_n$

Momentum: $\quad A_k^\dagger A_k \longleftrightarrow \ldots \neq C_k^\dagger C_k$

These properties are different for hard core bosons and free fermions.

- For an infinite, homogeneous lattice at T=0:





In some aspects, they behave like free fermions.
In other aspects, they are totally different.

Quasi-momentum distribution:

$$\langle A_k^\dagger A_k \rangle = \mathrm{DFT}[\underbrace{\langle a_n^\dagger a_m \rangle_{\rho_B}}]$$

$$\langle c_m^\dagger (-1)^{\sum_{n<l<m} c_l^\dagger c_l} c_n \rangle_{\rho_F} \xrightarrow[\text{Wick's theorem}]{} \sum \prod \langle c_\alpha^\dagger c_\beta \rangle_{\rho_F} = \det[\langle c_\alpha^\dagger c_\beta \rangle_{\rho_F}]$$

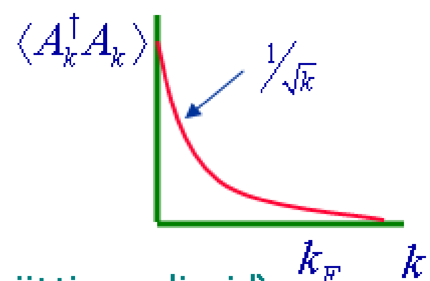
Töplitz determinant

small systems

Numerically:
Determinant of
$L \times L$ matrix
(efficient)

$|\alpha - \beta| \gg 1$

$$1 \Big/ \sqrt{|\alpha - \beta|}$$



(Shortcut: Lüttinger liquid)

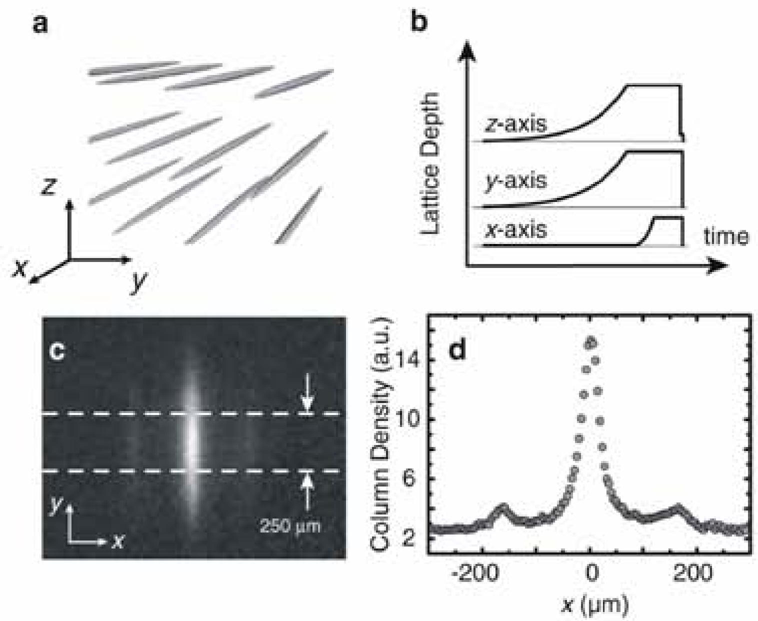
Ground-state wavefunction:

$$\Psi_B(x_1, x_2, \ldots, x_N) = \langle x_1, x_2, \ldots, x_N \mid \Psi \rangle = |\Psi_F(x_1, x_2, \ldots, x_N)| = \prod_{i<j} |\sin\left(\pi(x_i - x_j)/L\right)|$$

$x = 1, \ldots, L$

Slater determinant

Comparison with experiment



Facts to be taken into account:

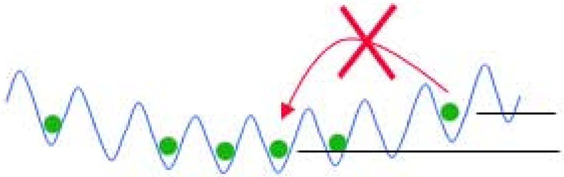
- External harmonic potential.
- Number of atoms is different in the tubes.
- Finite temperatures.

Other issues:

- Is the momentum distribution affected during expansion?

External harmonic potential:

- Extra condition for the JW mapping.

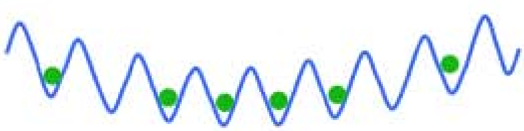


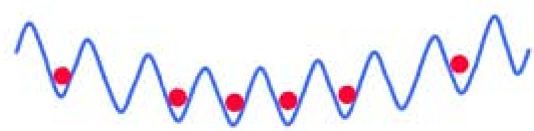
$\frac{1}{2} m\omega^2 R^2 \square\; U$

Hamiltonian:

$$H_B = -t\sum_n \left(a_{n+1}^\dagger a_n + a_n^\dagger a_{n+1}\right) + b\sum_n n^2 a_n^\dagger a_n \longleftrightarrow H_F = -t\sum_n \left(c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1}\right) + b\sum_n n^2 c_n^\dagger c_n$$

Density operator: $\rho_B \square\; e^{\beta H_B} \longleftrightarrow \rho_F \square\; e^{\beta H_F}$





TONKS-GIRARDEAU GAS

In the gas phase: Olshanii et al., Cazalilla, Giorgini et al, Drummond et al, Shlyapnikov et al...

Some properties coincide with those of free fermions:

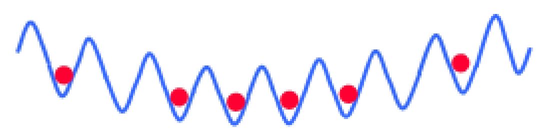
Density: $n(x) \overset{TF}{\approx} \frac{1}{\pi} \arccos\left(\max\left[\frac{\mu - bx^2}{-2t}, -1 \right] \right)$ $\Rightarrow$ size of the cloud R $\Rightarrow$ condition $\frac{1}{2} m\omega^2 R^2 \approx U$

Energy, excitation frequencies, entropy, etc.

Other properties are different:

Quasi-momentum distribution:

$$\langle A_k^\dagger A_k \rangle = \mathrm{DFT}[\underbrace{\langle a_n^\dagger a_m \rangle_{\rho_B}}_{\langle c_m^\dagger (-1)^{\sum_{n<l<m} c_l^\dagger c_l} c_n \rangle_{\rho_F}}]$$



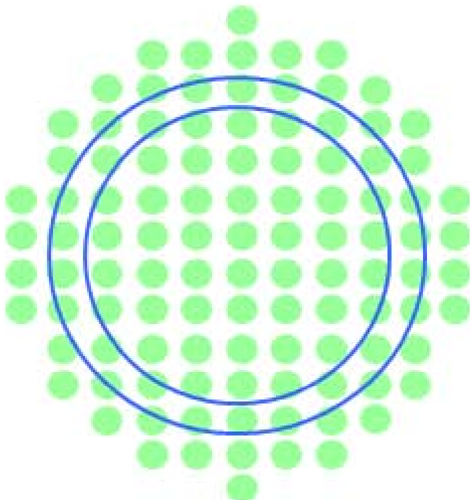
$$\langle c_m^\dagger (-1)^{\sum_{n<l<m} c_l^\dagger c_l} c_n \rangle_{\rho_F} \xrightarrow[\text{theorem}]{\text{Wick's}} \sum \prod \langle c_\alpha^\dagger c_\beta \rangle_{\rho_F} = \det[\langle c_\alpha^\dagger c_\beta \rangle_{\rho_F}]$$

Töplitz determinant

This can be efficiently evaluated

Number distribution in the 1D tubes:

At some point the tubes get isolated and the number of atoms remains fixed.



2D Harmonic trap

$$E = \sum_{\text{tubes}} E_t^{1D}(N_t) + N_t V_t - \mu N_t$$

1D Thomas-Fermi

Lagrange multiplier

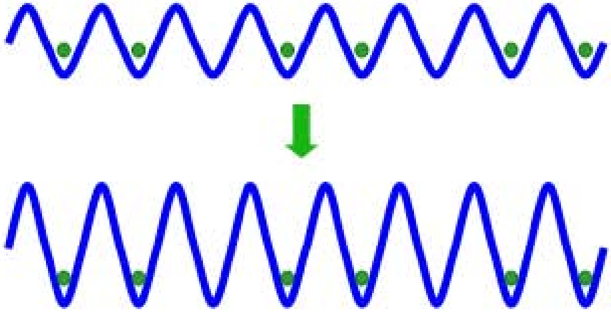
$$\frac{\partial E}{\partial N_t} = 0 \quad \Rightarrow P(N) = \frac{K}{N^{1/3}}$$

Normalization $\int_0^{N_0} P(N)dN = 1$

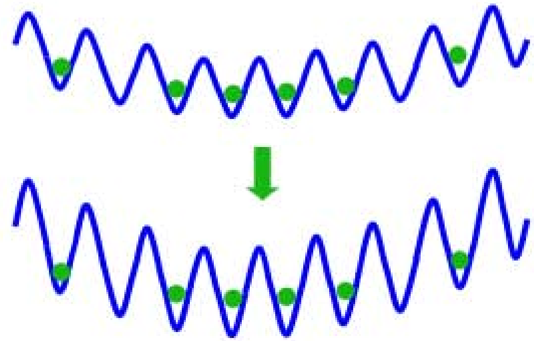
The distribution is completely fixed by N_0

Fitting parameter

Finite temperature:



$t/T = \text{const.}$



-Eigenfunctions remain invariant.
- Energies are proportional to t:

$$H_F = -t\sum_n \left(c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1}\right)$$

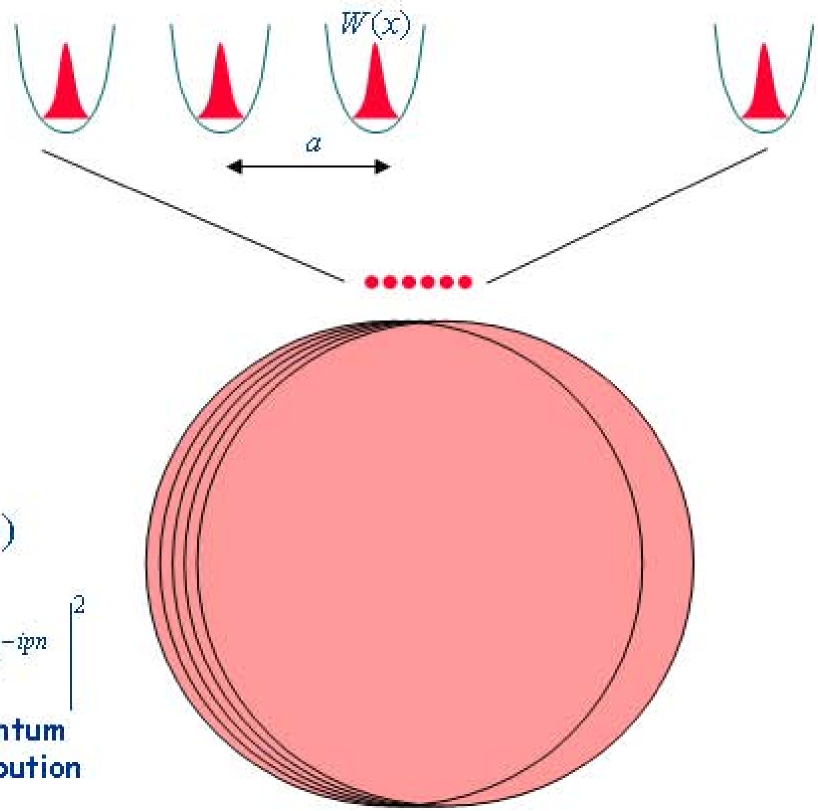
-The process is adiabatic.
- The entropy remains constant.
- The entropy is that of free fermions

We fit the initial temperature T_0

Expansion:

Ideal case:

$$\Psi(x,0)=\sum_n c_n\ W(x-an)$$



$$\Psi(x,t)=\sum_n c_n\ W(x-an,t)$$

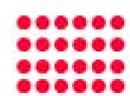
$$\Psi(x=p\,{\hbar t}/{ma},t)\xrightarrow{t\to\infty}\sum_n c_n e^{-ipn}\ \tilde{W}(p)$$

$$n(x=p\,{\hbar t}/{ma},t)\xrightarrow{t\to\infty}\underbrace{|\tilde{W}(p)|^2}_{\text{envelope}}\ \underbrace{\left|\sum_n c_n e^{-ipn}\right|^2}_{\text{Momentum distribution}}$$

Effects of interactions:

- The envelope is practically unaffected.
- The phase of the coefficients may be affected.

Taking Gaussian Wannier function, this can be easily estimated.



Change of the phase of a coefficient due to the interactions with the rest of the atoms <0.02

Momentum distribution:

$$\hat{n}_{M_k,T_k}(p) = \left|\tilde{W}(p)\right|^2 \sum_{\ell,\ell'=-\infty}^{\infty} e^{-ip(\ell-\ell')} \left\langle a_\ell^\dagger a_{\ell'} \right\rangle_{\rho_B}$$

J-W Transformation

$$\left\langle c_\ell^\dagger (-1)^{\sum_{\ell > m > \ell'} c_m^\dagger c_m} c_{\ell'} \right\rangle_{\rho_F} \quad \text{with} \quad \rho_F \propto e^{-\beta H - \mu N}$$

W. Theorem

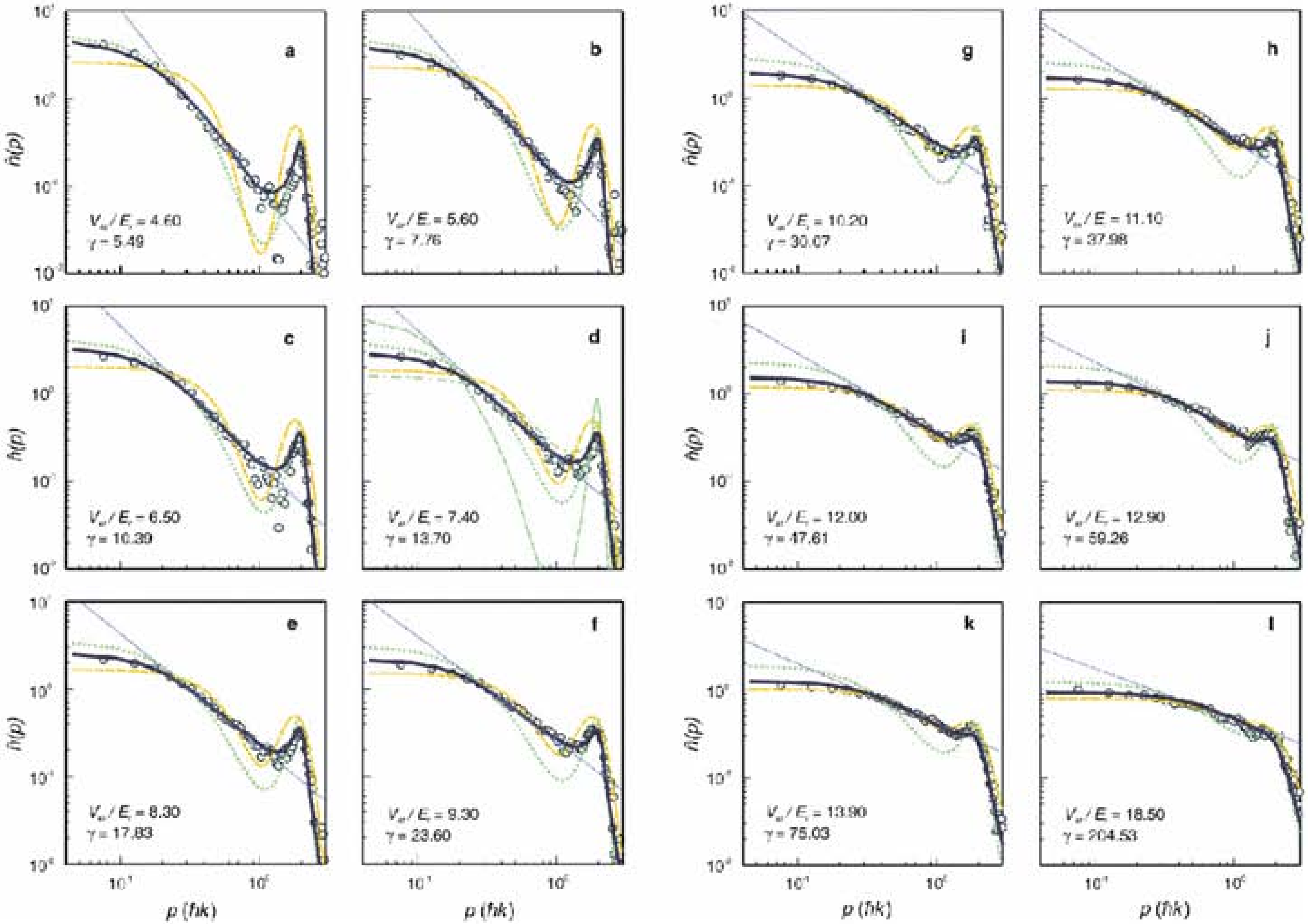
free Fermions

$|\ell-\ell'| \times |\ell-\ell'|$ Töplitz determinant in terms of $\left\langle c_k^\dagger c_{k'} \right\rangle_{\rho_F}$

We have to average with respect to the tubes:

- Number of atoms.
- Temperature.

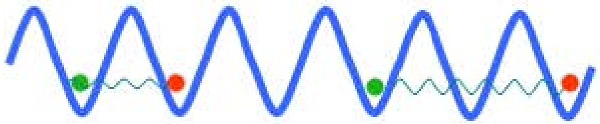
$$\hat{n}(p) = \int_0^{N_0} P(M)\hat{n}_{M,T_M}(p)\,dM$$

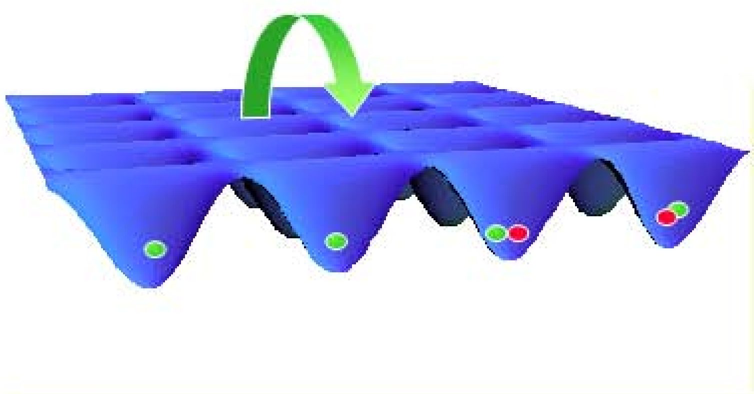


Fitting parameters: N_0, T_0

2. Extensions

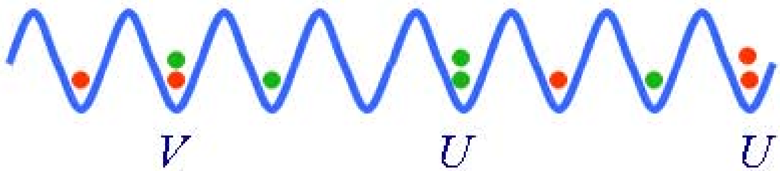
B. Paredes and G. Shlyapnikov

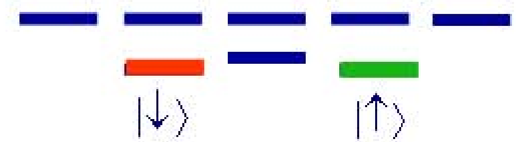




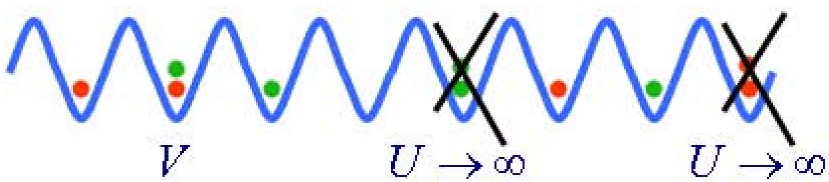
2.1. Two-component Bosons

(B. Paredes and J.I. Cirac, 2003)





- We will consider:
  - Strong interacting regime: $t, |V| \ll U$
  - Low temperature: $k_B T \ll U$
  - Dilute: $\nu_i = {N_i}/{L} < 1$



may behave, in some aspects, as interacting spin ½ Fermions

JORDAN-WIGNER TRANSFORMATION

Hard-core Bosons		**Fermions**
OPERATORS (OBSERVABLES)	$\longleftrightarrow$	OPERATORS (OBSERVABLES)
$a_{n,\sigma}$		$(-1)^{\sum\limits_{m<n} c^{\dagger}_{m,\sigma} c_{m,\sigma}} c_{n,\sigma}$

Hamiltonian:

$$H_B = -t\sum_{n,\sigma}\left(a^{\dagger}_{n+1,\sigma}a_{n,\sigma} + a^{\dagger}_{n,\sigma}a_{n+1,\sigma}\right) + \sum_{n,\sigma} E_{n,\sigma}a^{\dagger}_{n,\sigma}a_{n,\sigma} + V\sum_{n} a^{\dagger}_{n,\uparrow}a_{n,\uparrow}a^{\dagger}_{n,\downarrow}a_{n,\downarrow}$$

$$\longleftrightarrow$$

$$H_F = -t\sum_{n,\sigma}\left(c^{\dagger}_{n+1,\sigma}c_{n,\sigma} + c^{\dagger}_{n,\sigma}c_{n+1,\sigma}\right) + \sum_{n,\sigma} E_{n}c^{\dagger}_{n,\sigma}c_{n,\sigma} + V\sum_{n} c^{\dagger}_{n,\uparrow}c_{n,\uparrow}c^{\dagger}_{n,\downarrow}c_{n,\downarrow}$$

Density operator: $\rho_B \propto e^{-\beta H_B}$ $\longleftrightarrow$ $\rho_F \propto e^{-\beta H_F}$

Strong interacting bosons at temperature T		Interacting fermions at temperature T

$\rho_B \propto e^{-\beta H_B}$		$\rho_F \propto e^{-\beta H_F}$

- Strong interacting bosons behave as interacting fermions in several aspects.

For example:

- Density-density correlations: $\langle n_{x+\Delta} n_x \rangle$
- Spin-spin correlations: $\langle S_{x+\Delta} S_x \rangle$

where

$$n_x = a^\dagger_{x,\uparrow} a_{x,\uparrow} + a^\dagger_{x,\downarrow} a_{x,\downarrow}$$

$$S_x = a^\dagger_{x,\uparrow} a_{x,\uparrow} - a^\dagger_{x,\downarrow} a_{x,\downarrow}$$

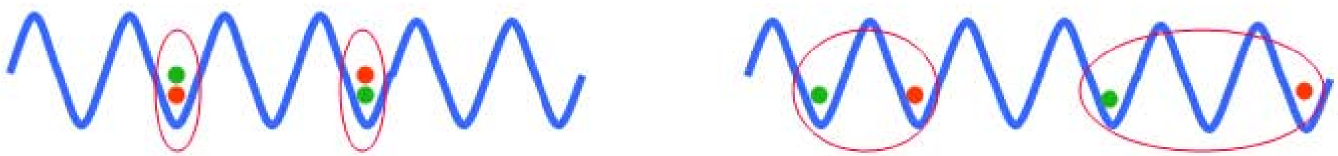
For V<0 these functions reflect the appearance of „Cooper“ pairs.

- In other aspects, the behavior will be different.

Let us consider $V < 0$

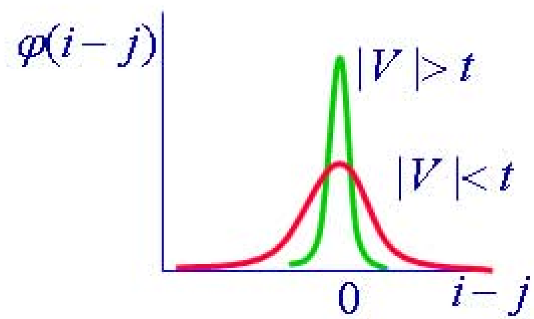
- Correlation functions: Cooper-pair formation
- Depending on the ratio $t/|V|$, we have localized/delocalized pairs.

$|V| > t$ $\qquad\qquad$ $|V| < t$

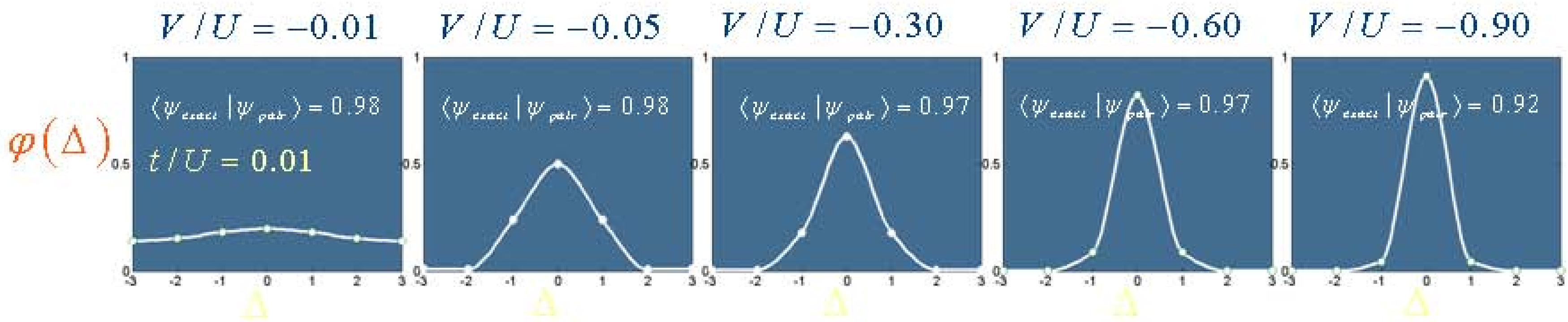


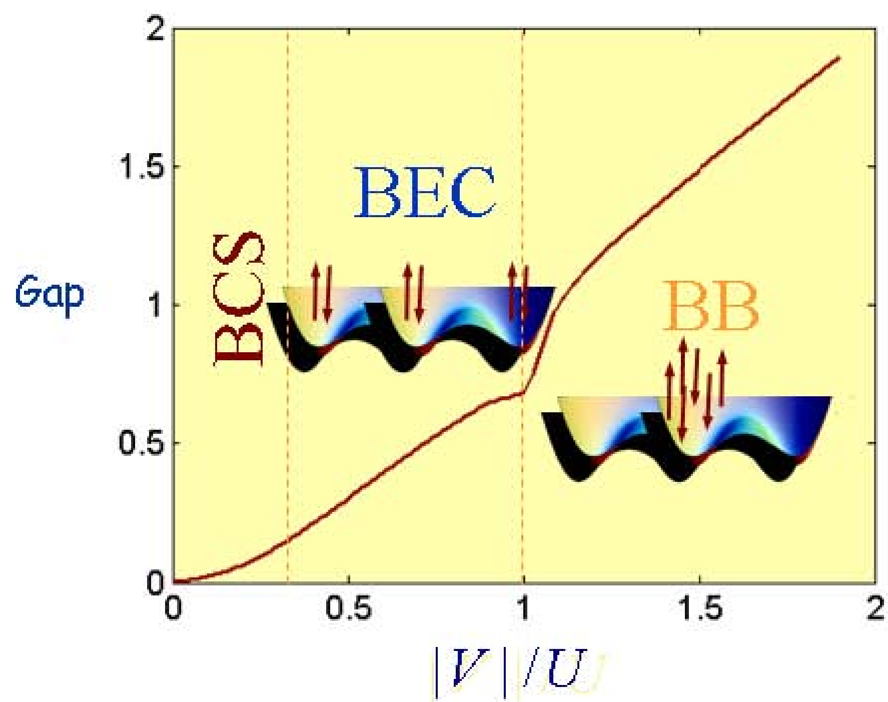
Qualitative behavior: BCS wavefunction

$$|\Psi_F\rangle \Box A\left\{\Pi\left[\sum_{i,j}\varphi(|i-j|)\,|i,j\rangle\right]^{\otimes N/2}\right\}$$

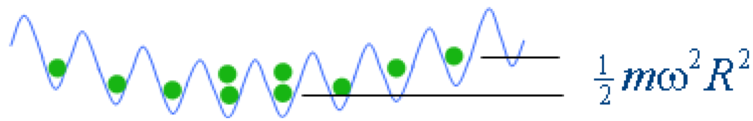


Numerical results



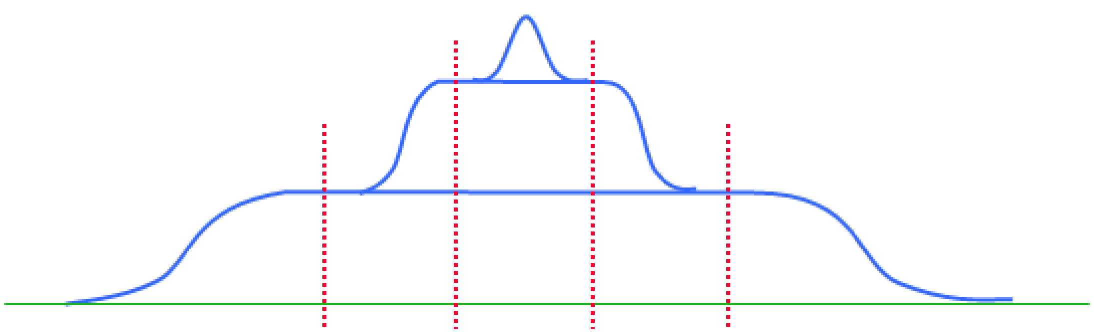


2.2. Higher filling factors:

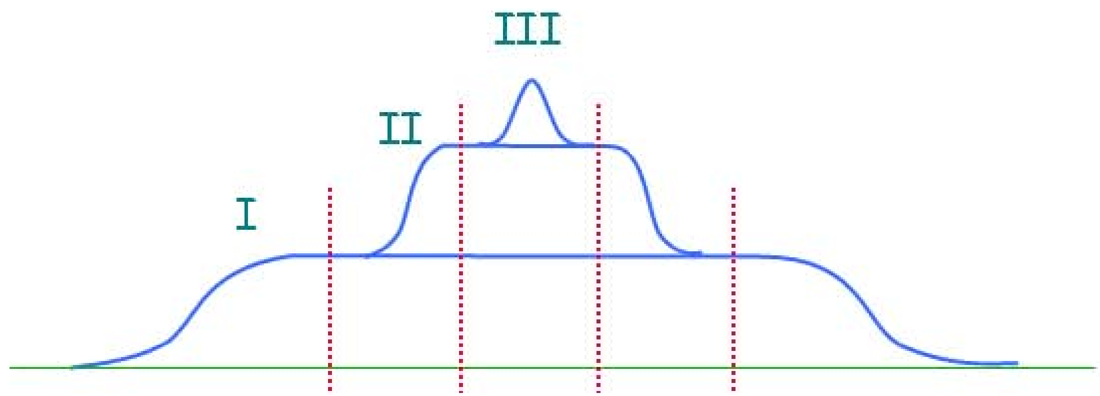


- Strong interacting regime: $t \ll U$
- Low temperature: $k_B T \ll U$
- Not-dilute: $\frac{1}{2} m\omega^2 R^2 \geq U$

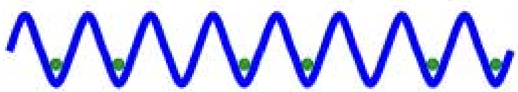
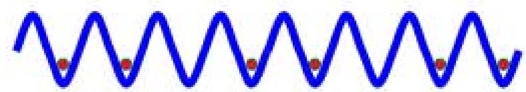
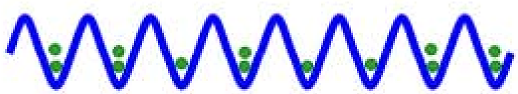
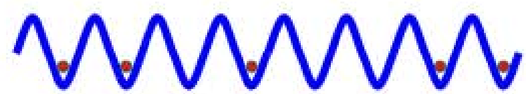
We have different layers of Tonks-Girardeau gases:



Description.

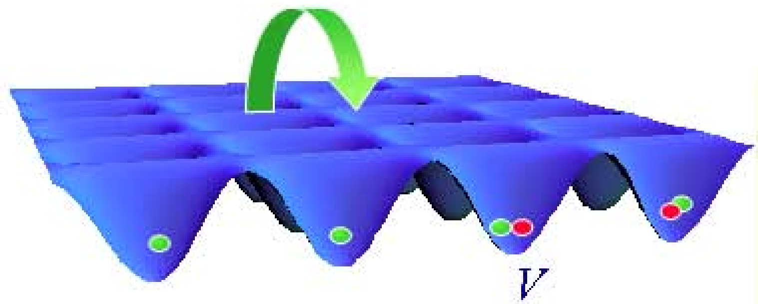


Fermionization:

	Hard-core Bosons	**Fermions**
Zone I:		
	$\mid 1,1,0,1,1,0,1,1\rangle$	$\mid 1,1,0,1,1,0,1,1\rangle$
	$-t\sum_n\left(a_{n+1}^{\dagger}a_n + a_n^{\dagger}a_{n+1}\right)$	$-t\sum_n\left(c_{n+1}^{\dagger}c_n + c_n^{\dagger}c_{n+1}\right)$
Zone II:		
	$\mid 2,2,1,2,2,1,2,2\rangle$	$\mid 1,1,0,1,1,0,1,1\rangle$
	$-t\sum_n\left(a_{n+1}^{\dagger}a_n + a_n^{\dagger}a_{n+1}\right)$	$-\sqrt{2}t\sum_n\left(c_{n+1}^{\dagger}c_n + c_n^{\dagger}c_{n+1}\right)$

The zones are determined variationally.

2.3. Higher dimensions



- Strong interacting regime: $t \ll U$
- Low temperature: $k_B T \ll U$
- Dilute: $\nu = N/L < 1$

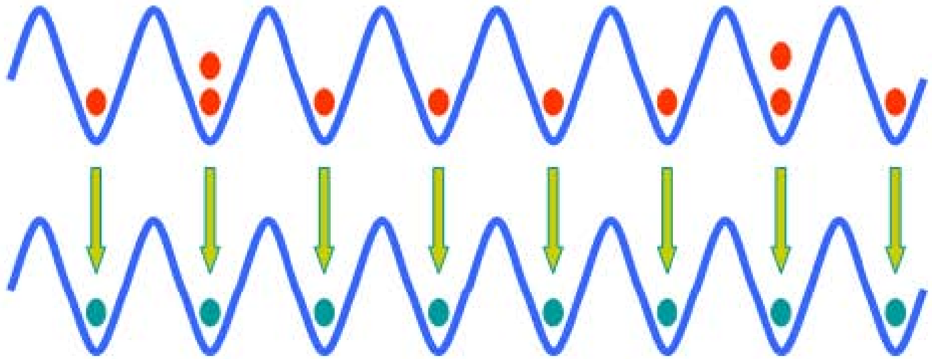
Still, the physics is similar but the fermionization procedure does not work.

- Cooper pairing.
- Analogon of high-Tc superconductivity.
- ???

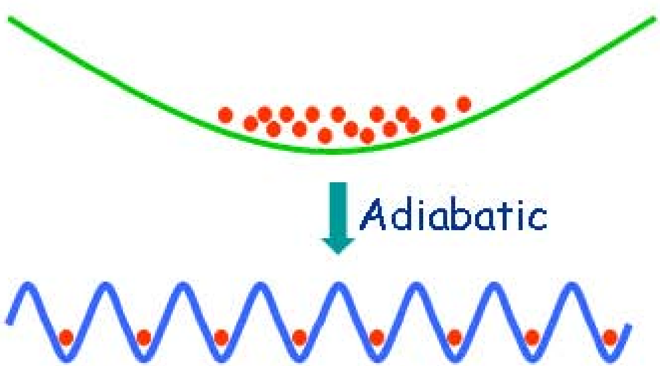
In 2D, in the weak interacting limit: M. Kagan et al.

3. Loading optical lattices

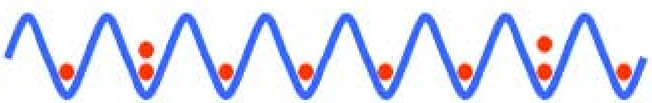
[P. Rabl, A.J. Daley, P.O. Fedichev, J.I. Cirac, and P. Zoller, 2003]



Problem: Trapping single atoms:



In reality:

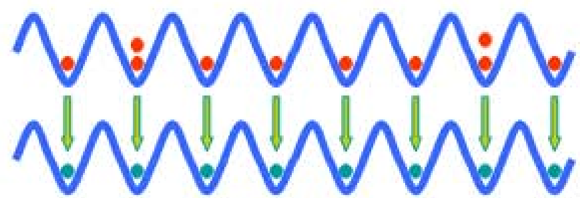


- Atoms/sites not commensurable.
- Temperature is finite.
- Residual potential.

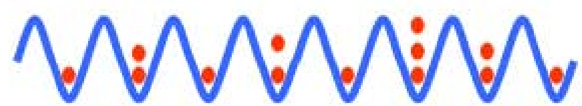
Solutions:

1. Lower the temperatures or count the atoms: Hard!

2. Coherently filter the state:



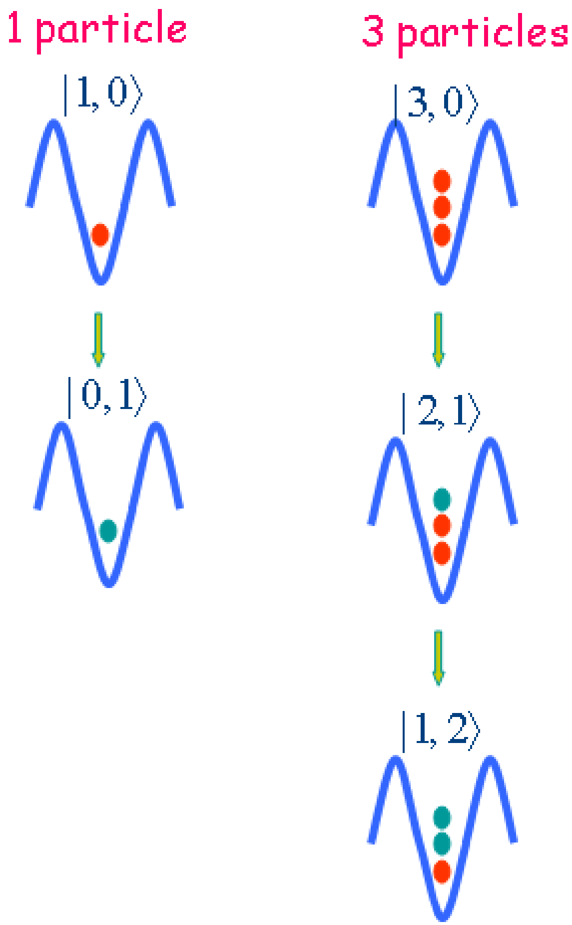
3. Redefine qubits and use new techniques:



Pattern loading

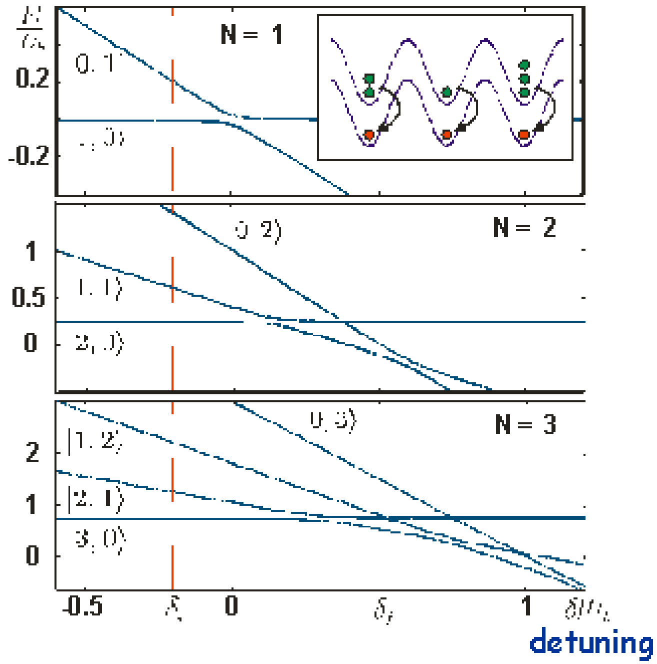
[P. Rabl, A.J. Daley, P.O. Fedichev, J.I. Cirac, and P. Zoller, 2003]

- Tunneling is switched off (raising the lattice potential). We can treat sites individually.
- Consider a particular site and two internal states:



Ideas:

- Interaction energy blocks transfer of more particles.
- Resonace frequency depends on the number of atoms, which is unknown.
- Use adiabatic passage to overcome this problem.



Conditions:

- $U_a \neq U_b$ since otherwise crossings take place at the wrong place.

  This can be achieved by having different potentials for a and b.

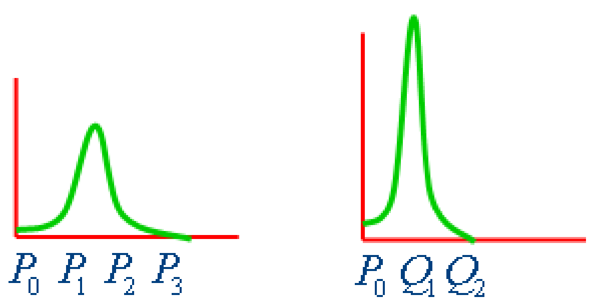
$U_a > U_b$

- Adiabaticity:

Results: atom-number distribution

Initially

Finally



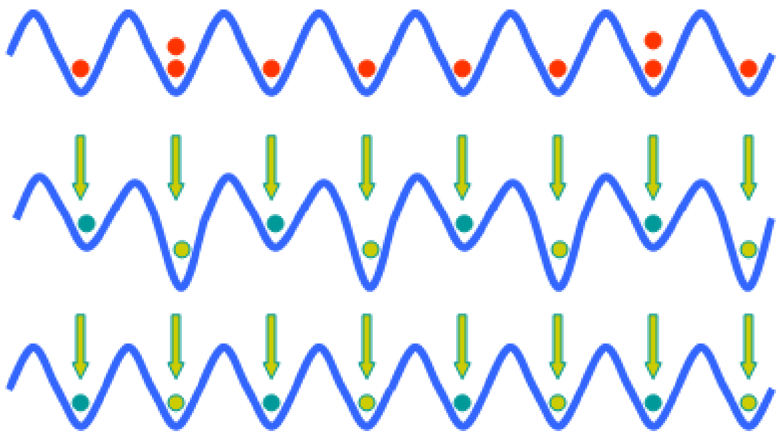
- Using a Grand-canonical distribution for the initial state with

  -Temperature such that there is 1 defect every 10 sites.

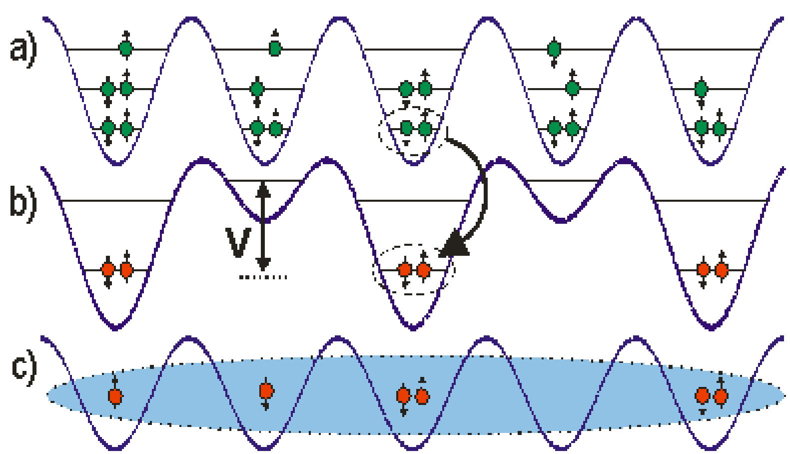
  - 2 atoms per well (in average).

one obtains 1 defect every 10.000 sites.

Also: quantum state engineering...



With Fermions: *Cooper pairs*

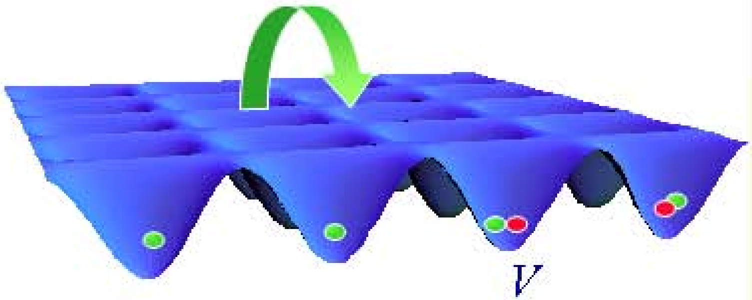


COLD ATOMS IN OPTICAL LATTICES

Fermions

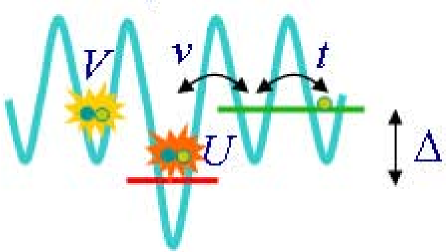
Hubbard model in 2D.

W. Hofstetter, J.I. Cirac, P. Zoller,
E. Demler, and M. Lukin



Anderson super-lattices.

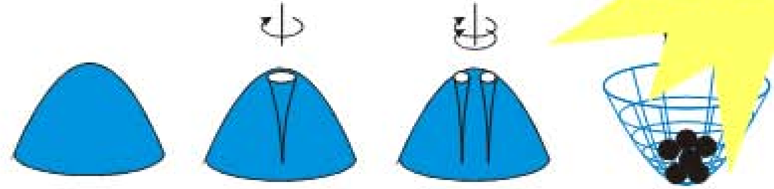
B. Paredes, C. Tejedor, and J.I. Cirac



Atoms in rotating traps.

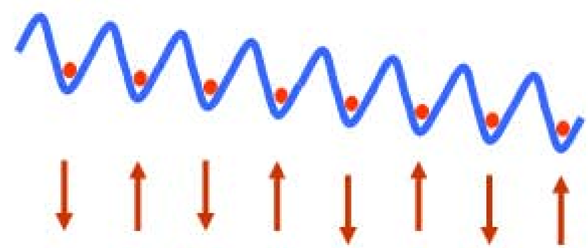
M. Popp, B. Paredes, and J.I. Cira

See poster by
Markus Popp



Spin systems.

J. Garcia-Ripoll, M.A. Martin-Delgado,
and J.I. Cirac

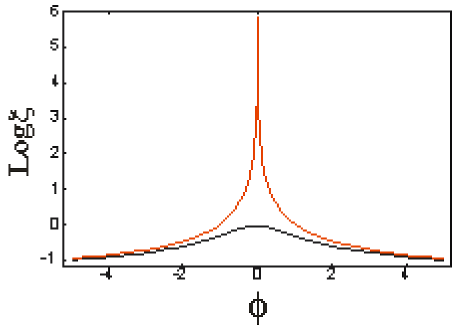


$$X_{k,k+1} \propto \lambda_1 \left[\left(\vec{S}_k \cdot \vec{S}_{k+1} \right) + \beta \left(\vec{S}_k \cdot \vec{S}_{k+1} \right)^2 \right] + (-1)^k \lambda_2 S_k^z$$

QUANTUM INFORMATION THEORY AND CONDENSED MATTER

Diverging entanglement length.

F. Verstraete, M.Popp, M.A. Martin-Delgado, and J.I. Cirac



Numerical simulation methods.

F. Verstraete, D. Porras, and J.I. Cirac