

Bose Einstein Condensates

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1 Introduction

In 1924 the Indian physicist S.N. Bose introduced a new statistical method to derive the black-body radiation law in terms of a gas of light quanta (photons). His work, together with the contemporary de Broglie's idea of matter-wave duality, led A. Einstein to apply the same statistical approach to a gas of N indistinguishable particles of mass m . An amazing result of his theory was the prediction that below some critical temperature a finite fraction of all the particles condense into the lowest energy single-particle state. This phenomenon, named Bose-Einstein condensation (BEC), is a consequence of purely statistical effects. For several years such a prediction received little attention, until 1938 when F. London argued that BEC could be at the basis of the superfluid properties observed in liquid ^4He below 2.17 K. A strong boost to the investigation of Bose-Einstein condensates was given in 1995 by the observation of BEC in dilute gases confined in magnetic traps and cooled down to temperatures of the order of a few nK. Differently

from superfluid helium, these gases allow one to tune the relevant parameters (confining potential, particle density, interactions, etc.), so to make them an ideal test-ground for concepts and theories on BEC.

2 What is BEC

In nature, particles have either integer or half-integer spin. Those having half-integer spin, like electrons, are called fermions and obey the Fermi-Dirac statistics; those having integer spin are called bosons and obey the Bose-Einstein statistics. Let us consider a system of N bosons. In order to introduce the concept of BEC on a general ground, one can start with the definition of the one-body density matrix

$$n^{(1)}(\mathbf{r}, \mathbf{r}') = \langle \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}') \rangle . \quad (1)$$

The quantities $\hat{\Psi}^\dagger(\mathbf{r})$ and $\hat{\Psi}(\mathbf{r})$ are the field operators which create and annihilate a particle at point \mathbf{r} , respectively; they satisfy the bosonic commutation relations

$$[\hat{\Psi}(\mathbf{r}), \hat{\Psi}^\dagger(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}') , \quad [\hat{\Psi}(\mathbf{r}), \hat{\Psi}(\mathbf{r}')] = 0 . \quad (2)$$

If the system is in a pure state described by the N -body wave function $\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N)$, then the average (1) is taken following the standard rules of quantum mechanics and the one-body density matrix can be written as

$$n^{(1)}(\mathbf{r}, \mathbf{r}') = N \int d\mathbf{r}_2 \dots d\mathbf{r}_N \Psi^*(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N) \Psi(\mathbf{r}', \mathbf{r}_2, \dots, \mathbf{r}_N) \quad (3)$$

involving the integration over the $N - 1$ variables $\mathbf{r}_2, \dots, \mathbf{r}_N$. In the more general case of a statistical mixture of pure states, expression (3) must be averaged according to the probability for a system to occupy the different states.

Since $n^{(1)}(\mathbf{r}, \mathbf{r}') = (n^{(1)}(\mathbf{r}', \mathbf{r}))^*$ the quantity $n^{(1)}$, when regarded as a matrix function of its indices \mathbf{r} and \mathbf{r}' , is Hermitian. It is therefore always possible to find a complete orthonormal basis of single-particle eigenfunctions, $\varphi_i(\mathbf{r})$, in terms of which the density matrix takes the diagonal form

$$n^{(1)}(\mathbf{r}, \mathbf{r}') = \sum_i n_i \varphi_i^*(\mathbf{r}) \varphi_i(\mathbf{r}') . \quad (4)$$

The real eigenvalues n_i are subject to the normalization condition $\sum_i n_i = N$ and have the meaning of occupation numbers of the single-particle states φ_i . BEC occurs when one of these numbers (say, n_0) becomes macroscopic, that is, when $n_0 \equiv N_0$ is a number of order N , all the others remaining of order 1. In this case Eq. (4) can be conveniently rewritten in the form

$$n^{(1)}(\mathbf{r}, \mathbf{r}') = N_0 \varphi_0^*(\mathbf{r}) \varphi_0(\mathbf{r}') + \sum_{i \neq 0} n_i \varphi_i^*(\mathbf{r}) \varphi_i(\mathbf{r}') \quad (5)$$

and the state represented by $\varphi_0(\mathbf{r})$ is called Bose-Einstein condensate. This definition is rather general, since it applies to any macroscopic ($N \gg 1$) system of indistinguishable bosons independently of mutual interactions and external fields.

The one-body density matrix (1) contains information on important physical observables. By setting $\mathbf{r} = \mathbf{r}'$ one finds the diagonal density of the system

$$n(\mathbf{r}) \equiv n^{(1)}(\mathbf{r}, \mathbf{r}) = \langle \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \rangle, \quad (6)$$

with $N = \int d\mathbf{r} n(\mathbf{r})$. The off-diagonal components can instead be used to calculate the momentum distribution

$$n(\mathbf{p}) = \langle \hat{\Psi}^\dagger(\mathbf{p}) \hat{\Psi}(\mathbf{p}) \rangle \quad (7)$$

where $\hat{\Psi}(\mathbf{p}) = (2\pi\hbar)^{-3/2} \int d\mathbf{r} \hat{\Psi}(\mathbf{r}) \exp[-i\mathbf{p} \cdot \mathbf{r}/\hbar]$ is the field operator in momentum representation. By inserting this expression for $\hat{\Psi}(\mathbf{p})$ into Eq. (7) one finds

$$n(\mathbf{p}) = \frac{1}{(2\pi\hbar)^3} \int d\mathbf{R} d\mathbf{s} n^{(1)}\left(\mathbf{R} + \frac{\mathbf{s}}{2}, \mathbf{R} - \frac{\mathbf{s}}{2}\right) e^{-i\mathbf{p} \cdot \mathbf{s}/\hbar}, \quad (8)$$

where $\mathbf{s} = \mathbf{r} - \mathbf{r}'$ and $\mathbf{R} = (\mathbf{r} + \mathbf{r}')/2$.

Let us consider a uniform system of N particles in a volume V and take the thermodynamic limit $N, V \rightarrow \infty$ with density N/V kept fixed. The eigenfunctions of the density matrix are plane waves and the lowest energy state has zero momentum, $\mathbf{p} = 0$, and constant wave function $\varphi_0(\mathbf{r}) = V^{-1/2}$. BEC in this state implies a macroscopic number of particles having zero momentum and constant density N_0/V . The density matrix only depends on $\mathbf{s} = \mathbf{r} - \mathbf{r}'$ and can be written as

$$n^{(1)}(\mathbf{s}) = \frac{N_0}{V} + \frac{1}{V} \sum_{\mathbf{p} \neq 0} n_{\mathbf{p}} e^{-i\mathbf{p} \cdot \mathbf{s}/\hbar}. \quad (9)$$

In the $s \rightarrow \infty$ limit, the sum on the right vanishes due to destructive interference between different plane waves, but the first term survives. One thus finds that, in the presence of BEC, the one-body density matrix tends to a constant finite value at large distances. This behavior is named *off-diagonal long range order*, since it involves the off-diagonal components of the density matrix. Its counterpart in momentum space is the appearance of a singular term at $\mathbf{p} = 0$:

$$n(\mathbf{p}) = N_0\delta(\mathbf{p}) + \sum_{\mathbf{p}' \neq 0} n_{\mathbf{p}'}\delta(\mathbf{p} - \mathbf{p}') . \quad (10)$$

The sum on the right is the number of noncondensed particles ($N - N_0$), and the quantity N_0/N is called condensate fraction.

If the system is not uniform the eigenfunctions of the density matrix are no longer plane waves but, provided N is sufficiently large, the concept of BEC is still well defined, being associated with the occurrence of a macroscopic occupation of a single-particle eigenfunction $\varphi_0(\mathbf{r})$ of the density matrix. Thus the condensed bosons can be described by means of the function $\Psi(\mathbf{r}) = \sqrt{N_0}\varphi_0(\mathbf{r})$, which is a classical complex field playing the role of an *order parameter*. This is the analogue of the classical limit of quantum electrodynamics where the electromagnetic field replaces the microscopic description of photons. The function Ψ may also depend on time and can be written as

$$\Psi(\mathbf{r}, t) = |\Psi(\mathbf{r}, t)|e^{iS(\mathbf{r}, t)} . \quad (11)$$

Its modulus determines the contribution of the condensate to the diagonal density (6), while the phase S is crucial in characterizing the coherence and superfluid properties of the system. The order parameter (11), also named *macroscopic wave function* or *condensate wave function*, is defined only up to a constant phase factor. One can always multiply this function by the numerical factor $e^{i\alpha}$ without changing any physical property. This reflects the gauge symmetry exhibited by all the physical equations of the problem. Making an explicit choice for the value of the order parameter, and hence for the phase, corresponds to a formal breaking of gauge symmetry.

3 BEC in ideal gases

Once we have defined what is a Bose-Einstein condensate, the next question is when such a condensation occurs in a given system. The ideal Bose gas

provides the simplest example. So, let us consider a gas of noninteracting bosons described by the Hamiltonian $\hat{H} = \sum_i \hat{H}_i^{(1)}$, where the Schrödinger equation $\hat{H}^{(1)}\varphi_i(\mathbf{r}) = \epsilon_i\varphi_i(\mathbf{r})$ gives the spectrum of single-particle wave functions and energies. One can define an occupation number n_i as the number of particles in the state with energy ϵ_i . Thus, any given state of the many-body system is specified by a set $\{n_i\}$. The mean occupation numbers, \bar{n}_i , can be calculated by using the standard rules of statistical mechanics. For instance, by considering a grand canonical ensemble at temperature T , one finds

$$\bar{n}_i = \{\exp[\beta(\epsilon_i - \mu)] - 1\}^{-1} \quad (12)$$

with $\beta = 1/(k_B T)$. The chemical potential μ is fixed by the normalization condition $\sum_i \bar{n}_i = N$, where N is the average number of particles in the gas. For $T \rightarrow \infty$ the chemical potential is negative and large. It increases monotonically when T is lowered. Let us call ϵ_0 the lowest single particle level in the spectrum. If at some critical temperature T_c the normalization condition can be satisfied with $\mu \rightarrow \epsilon_0^-$, then the occupation of the lowest state, $\bar{n}_0 = N_0$, becomes of order N and BEC is realized. Below T_c the normalization condition must be replaced with $N = N_0 + N_T$, where $N_T = \sum_{i \neq 0} \bar{n}_i$ is the number of particles out of the condensate, i.e., the *thermal* component of the gas. Whether BEC occurs or not, and what is the value of T_c depends on the dimensionality of the system and the type of single particle spectrum.

The simplest case is that of a gas confined in a cubic box of volume $V = L^3$ with periodic boundary conditions, where $\hat{H}^{(1)} = -(\hbar^2/2m)\nabla^2$. The eigenfunctions are plane waves $\varphi_{\mathbf{p}}(\mathbf{r}) = V^{-1/2} \exp[-i\mathbf{p} \cdot \mathbf{r}/\hbar]$ with energy $\epsilon_p = p^2/2m$ and momentum $\mathbf{p} = 2\pi\hbar\mathbf{n}/L$. Here \mathbf{n} is a vector whose components n_x, n_y, n_z are 0 or \pm integers. The lowest eigenvalue has zero energy ($\epsilon_0 = 0$) and zero momentum. The mean occupation numbers are given by $\bar{n}_{\mathbf{p}} = \{\exp[\beta(p^2/2m - \mu)] - 1\}^{-1}$. In the thermodynamic limit ($N, V \rightarrow \infty$ with N/V kept constant) one can replace the sum $\sum_{\mathbf{p}}$ with the integral $\int d\epsilon \rho(\epsilon)$, where $\rho(\epsilon) = (2\pi)^{-2} V (2m/\hbar^2)^{3/2} \sqrt{\epsilon}$ is the density of states. In this way one can calculate the thermal component of the gas as a function of T , finding the critical temperature

$$k_B T_c = \frac{2\pi\hbar^2}{m} \left(\frac{N}{V\zeta(3/2)} \right)^{2/3} \quad (13)$$

where ζ is the Riemann zeta function and $\zeta(3/2) \simeq 2.612$. For $T > T_c$ one has $\mu < 0$ and $N_T = N$. For $T < T_c$ one instead has $\mu = 0$, $N_T = N - N_0$

and

$$N_0(T) = N[1 - (T/T_c)^{3/2}] . \quad (14)$$

The critical temperature turns out to be fully determined by the density N/V and by the mass of the constituents. These results were first obtained by A. Einstein in his seminal paper and used by F. London in the context of superfluid helium. We notice that the replacement of the sum with an integral in the above derivation is justified only if the thermal energy $k_B T$ is much larger than the energy spacing between single-particle levels, i.e., if $k_B T \gg h^2/2mV^{2/3}$. It is also worth noticing that the above expression for T_c can be written as $\lambda_T^3 N/V \simeq 2.612$, where $\lambda_T = [2\pi\hbar^2/(mk_B T)]^{1/2}$ is the thermal de Broglie wavelength. This is equivalent to say that BEC occurs when the mean distance between bosons is of the order of their de Broglie wavelength.

Another interesting case, which is relevant for the recent experiments with BEC in dilute gases confined in magnetic and/or optical traps, is that of an ideal gas subject to harmonic potentials. Let us consider, for simplicity, an isotropic external potential $V_{\text{ext}}(r) = (1/2)m\omega_{\text{ho}}^2 r^2$. The single-particle Hamiltonian is $\hat{H}^{(1)} = -(\hbar^2/2m)\nabla^2 + V_{\text{ext}}(r)$ and its eigenvalues are $\epsilon_{n_x, n_y, n_z} = (n_x + n_y + n_z + 3/2)\hbar\omega_{\text{ho}}$. The corresponding density of states is $\rho(\epsilon) = (1/2)(\hbar\omega_{\text{ho}})^{-3}\epsilon^2$. A natural thermodynamic limit for this system is obtained by letting $N \rightarrow \infty$ and $\omega_{\text{ho}} \rightarrow 0$, while keeping the product $N\omega_{\text{ho}}^3$ constant. The condition for BEC to occur is that μ approaches the value $\epsilon_{000} = (3/2)\hbar\omega_{\text{ho}}$ from below by cooling the gas down to T_c . Following the same procedure as for the uniform gas, one finds

$$k_B T_c = \hbar\omega_{\text{ho}}[N/\zeta(3)]^{1/3} = 0.94 \hbar\omega_{\text{ho}} N^{1/3} . \quad (15)$$

and

$$N_0(T) = N[1 - (T/T_c)^3] . \quad (16)$$

Notice that the condensate is not uniform in this case, since it corresponds to the lowest eigenfunction of the harmonic oscillator, which is a Gaussian of width $a_{\text{ho}} = [\hbar/(m\omega_{\text{ho}})]^{1/2}$. Correspondingly, the condensate in the momentum space is also a Gaussian, of width a_{ho}^{-1} . This implies that, differently from the gas in a box, here the condensate can be seen both in coordinate and momentum space in the form of a narrow distribution emerging from a wider thermal component. Finally, results (15) and (16) remain valid even for anisotropic harmonic potentials, with trapping frequencies ω_x , ω_y and ω_z , provided the frequency ω_{ho} is replaced by the geometric average $(\omega_x\omega_y\omega_z)^{1/3}$.

4 BEC in interacting gases

Actual condensates are made of interacting particles. The full many-body Hamiltonian is

$$\begin{aligned} \hat{H} &= \int d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) \hat{H}_0 \hat{\Psi}(\mathbf{r}) \\ &+ \frac{1}{2} \int d\mathbf{r}' d\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}^\dagger(\mathbf{r}') V(\mathbf{r} - \mathbf{r}') \hat{\Psi}(\mathbf{r}') \hat{\Psi}(\mathbf{r}) , \end{aligned} \quad (17)$$

where $V(\mathbf{r} - \mathbf{r}')$ is the particle-particle interaction and $\hat{H}_0 = -(\hbar^2/2m)\nabla^2 + V_{\text{ext}}(\mathbf{r})$. Differently from the case of ideal gases, \hat{H} is no longer a sum of single-particle Hamiltonians. However, the general definitions given in section 2 are still valid. In particular, the one-body density matrix, in the presence of BEC, can be separated as in Eq. (5). One can write $n^{(1)}(\mathbf{r}, \mathbf{r}') = \Psi^*(\mathbf{r})\Psi(\mathbf{r}') + \tilde{n}^{(1)}(\mathbf{r}, \mathbf{r}')$, where Ψ is the order parameter of the condensate ($\Psi^*(\mathbf{r})\Psi(\mathbf{r}')$ being of order N), while $\tilde{n}^{(1)}(\mathbf{r}, \mathbf{r}')$ vanishes for large $|\mathbf{r} - \mathbf{r}'|$. This is equivalent to say that the bosonic field operator splits in two parts,

$$\hat{\Psi}(\mathbf{r}) = \Psi(\mathbf{r}) + \delta\hat{\Psi}(\mathbf{r}) , \quad (18)$$

where the first term is a complex function and the second one is the field operator associated with the noncondensed particles. This decomposition is particularly useful when the depletion of the condensate, i.e., the fraction of noncondensed particles, is small. This happens when the interaction is weak, but also for particles with arbitrary interaction, provided the gas is dilute. In this case, one can expand the many-body Hamiltonian by treating the operator $\delta\hat{\Psi}$ as a small quantity.

A suitable strategy consists in writing the Heisenberg equation for the evolution of the field operators, $i\hbar\partial_t\hat{\Psi} = [\hat{\Psi}, \hat{H}]$, using the many-body Hamiltonian (17):

$$i\hbar\partial_t\hat{\Psi}(\mathbf{r}, t) = \left(\hat{H}_0 + \int d\mathbf{r}' \hat{\Psi}^\dagger(\mathbf{r}', t) V(\mathbf{r} - \mathbf{r}') \hat{\Psi}(\mathbf{r}', t) \right) \hat{\Psi}(\mathbf{r}, t) . \quad (19)$$

The zeroth-order is thus obtained by replacing the operator $\hat{\Psi}$ with the classical field Ψ . In the integral containing the interaction $V(\mathbf{r} - \mathbf{r}')$, this replacement is, in general, a poor approximation when short distances ($\mathbf{r} - \mathbf{r}'$) are involved. In a dilute and cold gas, one can nevertheless obtain a proper expression for the interaction term by observing that, in this case, only binary

collisions at low energy are relevant and these collisions are characterized by a single parameter, the s -wave scattering length, a , independently of the details of the two-body potential. This allows one to replace $V(\mathbf{r} - \mathbf{r}')$ in \hat{H} with an effective interaction $V(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}')$ where the coupling constant g is given by $g = 4\pi\hbar^2 a/m$. The scattering length can be measured with several experimental techniques or calculated from the exact two-body potential. Using this pseudo-potential and replacing the operator $\hat{\Psi}$ with the complex function Ψ in the Heisenberg equation of motion, one gets

$$i\hbar\partial_t\Psi(\mathbf{r}, t) = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) + g|\Psi(\mathbf{r}, t)|^2 \right) \Psi(\mathbf{r}, t). \quad (20)$$

This is known as Gross-Pitaevskii (GP) equation and it was first introduced in 1961. It has the form of a *nonlinear Schrödinger equation*, the nonlinearity coming from the mean-field term, proportional to $|\Psi|^2$. It has been derived assuming that N is large while the fraction of noncondensed atoms is negligible. On the one hand, this means that quantum fluctuations of the field operator have to be small, which is true when $n|a|^3 \ll 1$, where n is the particle density. In fact, one can show that, at $T = 0$ the quantum depletion of the condensate is proportional to $(n|a|^3)^{1/2}$. On the other hand, thermal fluctuations have also to be negligible and this means that the theory is limited to temperatures much lower than T_c . Within these limits, one can identify the total density with the condensate density.

The stationary solution of Eq. (20) corresponds to the condensate wave function in the ground state. One can write $\Psi(\mathbf{r}, t) = \Psi_0(\mathbf{r}) \exp(-i\mu t/\hbar)$, where μ is the chemical potential. Then the GP equation (20) becomes

$$\left(-\frac{\hbar^2\nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) + g|\Psi_0(\mathbf{r})|^2 \right) \Psi_0(\mathbf{r}) = \mu\Psi_0(\mathbf{r}). \quad (21)$$

where $n(\mathbf{r}) = |\Psi_0(\mathbf{r})|^2$ is the particle density. The same equation can be obtained by minimizing the energy of the system written as a functional of the density:

$$E[n] = \int d\mathbf{r} \left[\frac{\hbar^2}{2m} |\nabla\sqrt{n}|^2 + nV_{\text{ext}}(\mathbf{r}) + \frac{gn^2}{2} \right]. \quad (22)$$

The first term corresponds to the quantum kinetic energy coming from the uncertainty principle; it is usually named “quantum pressure” and vanishes for uniform systems.

The next order in $\delta\hat{\Psi}$ gives the excited states of the condensate. In a uniform gas the ground state order parameter, Ψ_0 , is a constant and the first order expansion of \hat{H} was introduced by N. Bogoliubov in 1947. In particular, he found an elegant way to diagonalize the Hamiltonian by using simple linear combinations of particle creation and annihilation operators. These are known as Bogoliubov's transformations and stay at the basis of the concept of *quasiparticle*, one of the most important concepts in quantum many-body theory.

A generalization of Bogoliubov's approach to the case of nonuniform condensates is obtained by considering small deviations around the ground state in the form

$$\Psi(\mathbf{r}, t) = e^{-i\mu t/\hbar} \left[\Psi_0(\mathbf{r}) + u(\mathbf{r})e^{-i\omega t} + v^*(\mathbf{r})e^{i\omega t} \right]. \quad (23)$$

Inserting this expression into Eq. (20) and keeping terms linear in the complex functions u and v , one gets

$$\hbar\omega u(\mathbf{r}) = [\hat{H}_0 - \mu + 2g\Psi_0^2(\mathbf{r})]u(\mathbf{r}) + g\Psi_0^2(\mathbf{r})v(\mathbf{r}) \quad (24)$$

$$-\hbar\omega v(\mathbf{r}) = [\hat{H}_0 - \mu + 2g\Psi_0^2(\mathbf{r})]v(\mathbf{r}) + g\Psi_0^2(\mathbf{r})u(\mathbf{r}). \quad (25)$$

These coupled equations allow one to calculate the energies $\varepsilon = \hbar\omega$ of the excitations. They also give the so-called quasiparticle amplitudes u and v , which obey the normalization condition $\int d\mathbf{r} [u_i^*(\mathbf{r})u_j(\mathbf{r}) - v_i^*(\mathbf{r})v_j(\mathbf{r})] = \delta_{ij}$. In a uniform gas, u and v are plane waves and one recovers the famous Bogoliubov's spectrum

$$\hbar\omega = \left[\frac{\hbar^2 q^2}{2m} \left(\frac{\hbar^2 q^2}{2m} + 2gn \right) \right]^{1/2} \quad (26)$$

where \mathbf{q} is the wavevector of the excitations. For large momenta the spectrum coincides with the free-particle energy $\hbar^2 q^2/2m$. At low momenta it instead gives the phonon dispersion $\omega = cq$, where $c = [gn/m]^{1/2}$ is the Bogoliubov sound velocity. The transition between the two regimes occurs when the excitation wavelength is of the order of the *healing length*,

$$\xi = [8\pi na]^{-1/2} = \hbar/(mc\sqrt{2}), \quad (27)$$

which is an important lengthscale for superfluidity. When the order parameter is forced to vanish at some point (by an impurity, a wall, etc.), the healing length provides the typical distance over which it recovers its bulk value. In

a nonuniform condensate the excitations are no longer plane waves but, at low energy, they have still a phonon-like character, in the sense that they involve a collective motion of the condensate.

The GP equation (20) is the starting point for an accurate mean-field description of BEC in dilute cold gases, which is rigorous at $T = 0$ and for $n|a|^3 \ll 1$. Static and dynamics properties of condensates in different geometries can be calculated by solving the GP equation numerically or using suitable approximated methods. The inclusion of effects beyond mean-field is a highly nontrivial and interesting problem. A rather extreme case is represented by liquid ^4He , which is a dense system where the interaction between atoms causes a large depletion of the condensate even at $T = 0$ (N_0/N being less than 10%) and thus a full many-body treatment is required for its rigorous description. Nevertheless, even in this case, the general definitions of section 2 are still useful.

5 Superfluidity and coherence

With the word superfluidity one summarizes a complex of macroscopic phenomena occurring in quantum fluids under particular conditions: persistent currents, equilibrium states at rest in rotating vessels, viscousless motion, quantized vorticity, and others. These features can be observed also in BEC. The link between BEC and superfluidity is given by the phase of the order parameter (11). To understand this point, let us consider a uniform system. If $\hat{\Psi}(\mathbf{r}, t)$ is a solution of the Heisenberg equation (19) with $V_{\text{ext}} = 0$, then

$$\hat{\Psi}'(\mathbf{r}, t) = \hat{\Psi}(\mathbf{r} - \mathbf{v}t, t) \exp \left[\frac{i}{\hbar} \left(m\mathbf{v} \cdot \mathbf{r} - \frac{1}{2}mv^2t \right) \right], \quad (28)$$

where \mathbf{v} is a constant vector, is also a solution. This equation gives the Galilean transformation of the field operator and applies also to its condensate component Ψ . At equilibrium, the ground state order parameter is given by $\Psi_0 = \sqrt{n} \exp(-i\mu t/\hbar)$, where n is a constant independent of \mathbf{r} . In a frame where the condensate moves with velocity \mathbf{v} , the order parameter instead takes the form $\Psi_0 = \sqrt{n} \exp(iS)$, with $S(\mathbf{r}, t) = \hbar^{-1}[m\mathbf{v} \cdot \mathbf{r} - (mv^2/2 + \mu)t]$. The velocity of the condensate can thus be identified with the gradient of the phase S :

$$\mathbf{v}(\mathbf{r}, t) = \frac{\hbar}{m} \nabla S(\mathbf{r}, t). \quad (29)$$

This definition is also valid for \mathbf{v} varying slowly in space and time. The modulus of the order parameter plays a minor role in this definition and it is not necessary to assume the gas to be dilute and close to $T = 0$. Indeed, the relation (29) between the velocity field and the phase of the order parameter applies also in the presence of large quantum depletion, as in superfluid ^4He , and at $T \neq 0$. In this case, n should not be identified with the condensate density. Conversely, in dilute gases at $T = 0$, n is the condensate density and the velocity (29) can be simply obtained by applying the usual definition of current density operator, $\hat{\mathbf{j}}$, to the order parameter (11).

The velocity (29) describes a potential flow and corresponds to a collective motion of many particles occupying a single quantum state. Being equal to the gradient of a scalar function, it is irrotational ($\nabla \times \mathbf{v}_s = 0$) and satisfies the Onsager-Feynman quantization condition $\oint \mathbf{v}_s \cdot d\mathbf{l} = \kappa h/m$, with κ non-negative integer. These conditions are not satisfied by a classical fluid, where the hydrodynamic velocity field, $\mathbf{v}(\mathbf{r}, t) = \mathbf{j}(\mathbf{r}, t)/n(\mathbf{r}, t)$, is the average over many different states and does not correspond to a potential flow.

By using the definition of the phase S and velocity \mathbf{v} , together with particle conservation, one can show that the dynamics of a condensate, as far as macroscopic motions are concerned, is governed by the hydrodynamic equations of an irrotational nonviscous fluid. Within mean-field theory, this can be easily seen by rewriting the GP equation (20) in terms of the density $n = |\Psi|^2$ and the velocity (29). Neglecting the quantum pressure term $\nabla^2 \sqrt{n}$ (hence limiting the description to lengthscales larger than the healing length ξ), one gets

$$\frac{\partial}{\partial t} n + \nabla \cdot (\mathbf{v}n) = 0 \quad (30)$$

and

$$m \frac{\partial}{\partial t} \mathbf{v} + \nabla \left(V_{\text{ext}} + \mu(n) + \frac{mv^2}{2} \right) = 0, \quad (31)$$

with the local chemical potential $\mu(n) = gn$. These equations have the typical structure of the dynamic equations of superfluids at zero temperature and can be viewed as the $T = 0$ case of the more general Landau's two-fluid theory.

One of the most striking evidences of superfluidity is the observation of quantized vortices, i.e., vortices obeying the Onsager-Feynman quantization condition. A vast literature is devoted to vortices in superfluid helium and,

more recently, vortices have been produced and studied also in condensates of ultracold gases, including nice configurations of many vortices in regular triangular lattices, similar to the Abrikosov lattices in superconductors. Other phenomena, such as the reduction of the moment of inertia, the occurrence of Josephson tunnelling through barriers, the existence of thresholds for dissipative processes (Landau criterion), and others, are typical subjects of intense investigation.

Another important consequence of the fact that BEC is described by an order parameter with a well defined phase is the occurrence of coherence effects which, in different words, mean that condensates behave like *matter waves*. For instance, one can measure the phase difference between two condensates by means of interference. This can be done in coordinate space by confining two condensates in two potential minima, a and b , at a distance d . Let us take d along z and assume that, at $t = 0$, the order parameter is given by the linear combination $\Psi(\mathbf{r}) = \Psi_a(\mathbf{r}) + \exp(i\phi)\Psi_b(\mathbf{r})$ with Ψ_a and Ψ_b real and without overlap. Then let us switch off the confining potentials so that the condensates expand and overlap. If the overlap occurs when the density is small enough to neglect interactions, the motion is ballistic and the phase of each condensate evolves as $S(\mathbf{r}, t) \simeq mr^2/(2\hbar t)$, so that $\mathbf{v} = \mathbf{r}/t$. This implies a relative phase $\phi + S(x, y, z + d/2) - S(x, y, z - d/2) = \phi + mdz/\hbar t$. The total density $n = |\Psi|^2$ thus exhibits periodic modulations along z with wavelength $\hbar t/md$. This interference pattern has indeed been observed in condensates of ultracold atoms. In these systems it was also possible to measure the coherence length, that is, the distance $|\mathbf{r} - \mathbf{r}'|$ at which the one-body density vanishes and the phase of the order parameter is no more well defined. In most situations, the coherence length turns out to be of the order of, or larger than the size of the condensates. However, interesting situations exist when the coherence length is shorter but the system still preserves some features of BEC (quasi-condensates).

6 Final remarks

Bose-Einstein condensates of ultracold atoms are easily manipulated by changing and tuning the external potentials. This means, for instance, that one can prepare condensates in different geometries, including very elongated (quasi-1D) or disk-shaped (quasi 2D) condensates. This is conceptually im-

portant, since BEC in lower dimensions is not as simple as in 3D: thermal and quantum fluctuations play a crucial role, superfluidity must be properly re-defined, and very interesting limiting cases can be explored (Tonks-Girardeau regime, Luttinger liquid, etc.). Another possibility is to use laser beams to produce standing waves acting as an external periodic potential (optical lattice). Condensates in optical lattices behave as a sort of perfect crystals, whose properties are the analog of the dynamic and transport properties in solid state physics, but with controllable spacing between sites, no defects and tunable lattice geometry. One can investigate the role of phase coherence in the lattice, looking for instance at Josephson effects as in a chain of junctions. By tuning the lattice depth one can explore the transition from a superfluid phase and a Mott-insulator phase, which is a nice example of quantum phase transition. Controlling cold atoms in optical lattice can be a good starting point for application in quantum engineering, interferometry and quantum information.

Another interesting aspect of BECs is that the key equation for their description in mean-field theory, namely the GP equation (20), is a nonlinear Schrödinger equation very similar to the ones commonly used, for instance, in nonlinear quantum optics. This opens interesting perspectives in exploiting the analogies between the two fields, such as the occurrence of dynamical and parametric instabilities, the possibility to create different types of solitons, the occurrence of nonlinear processes like, for example, higher harmonic generation and mode mixing.

A relevant part of the current research also involves systems made of mixtures of different gases, Bose-Bose or Fermi-Bose, and many activities with ultracold atoms now involve fermionic gases, where BEC can be also realized by condensing molecules of fermionic pairs. An extremely active research now concerns the BCS-BEC crossover, which can be obtained in Fermi gases by tuning the scattering length (and hence the interaction) by means of Feshbach resonances.

Ten years after the first observation of BEC in ultracold gases, it is almost impossible to summarize all the researches done in this field. A huge amount of work has been already devoted to characterize the condensates and several new lines have been opened. Rather detailed review articles and books are already available for the interested readers.

See also

Superfluidity

Equilibrium statistical mechanics

Interacting particle systems and hydrodynamic equations

Quantum phase transitions

Keywords

Ultracold gases

Superfluidity

Quantum statistics

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