Dynamics of two and three component Fermi mixtures

Päivi Törmä
Helsinki University of Technology

Cold Quantum Matter (EuroQUAM)
Fermix 2009 Meeting
5th June 2009
Trento, Italy

Funding: ESF (EuroQUAM, EURYI), Academy of Finland
Contents

• Three-component mixtures, attractive interactions: a BdG study in a trap (preliminary)
• The repulsive Hubbard model in a one-dimensional lattice: exact hopping modulation dynamics
• RF-spectroscopy in a 1D system: signatures of the FFLO state
• Fermi condesates as sensors
• Three-component mixtures, attractive interactions: a BdG study in a trap (preliminary)

• The repulsive Hubbard model in a one-dimensional lattice: exact hopping modulation dynamics

• RF-spectroscopy in a 1D system: signatures of the FFLO state

• Fermi condensates as sensors
Three-component system in a harmonic trap

• Two-component systems in harmonic traps described by Bogoliubov – deGennes (BdG) equations (e.g. Griffin, Törmä, Machida, Levin, Randeria, Stringari groups)

• BdG equations for 3 components (1, 2, 3): to allow detailed description of trapping effects

• 1+2 interacting, 2+3 interacting, 1+3 not (to simplify)

• Expansion into harmonic trap eigenstates

• Self-consistent equations

• Hartree fields ignored
The mean-field Hamiltonian:

$$H_{MF} = \sum_{kk'} \left[ \sum_\sigma \epsilon_{\sigma kk'} c_{\sigma k'}^\dagger c_{\sigma k} + \frac{1}{2} \sum_{\sigma \neq \sigma'} \left( J_{\sigma \sigma' kk'} c_{\sigma k'}^\dagger c_{\sigma' k} + J_{\sigma' \sigma kk'} c_{\sigma k'}^\dagger c_{\sigma' k} \right) + \frac{1}{2} \sum_{\sigma \neq \sigma'} \left( F_{\sigma \sigma' kk'}^* c_{\sigma k'}^\dagger c_{\sigma k} + F_{\sigma' \sigma kk'} c_{\sigma k'}^\dagger c_{\sigma k} \right) \right] + C.$$  

$k$: trap quantum numbers

Here $\epsilon_{\sigma kk'}$ is the single particle energy, integrated pairing potential $J_{\sigma \sigma' kk'}$, integrated Hartree potential and $C$ a constant shift in the energy.

O. Nummi, J. Kinnunen, P. Törmä, in preparation
MF-Hamiltonian in matrix form

\[ H_{MF} = c_{k'}^\dagger H_{kk'} c_k + C', \quad c_k^T = [c_{ak}, c_{bk}^\dagger, c_{ck}] \]

Bogoliubov transformation into quasiparticles

\[
\begin{pmatrix}
\gamma_{ak} \\
\gamma_{bk}^\dagger \\
\gamma_{ck}
\end{pmatrix} = B^k
\begin{pmatrix}
c_{ak} \\
c_{bk}^\dagger \\
c_{ck}
\end{pmatrix}
\]

\( B^k \) is unitary
Expand in harmonic trap eigenstates

$$
\Psi_\sigma(r) = \sum_{nlm} R_{nl}^\sigma(r) Y_{lm}(\Omega) c_{nlm\sigma}
$$

Separate different l-quantum numbers and get self-consistent equations for gaps \( \Delta_{12}, \Delta_{23} \) and densities \( n_1, n_2, n_3 \)
Example gap profiles

A smooth transition between two gaps and FFLO-type oscillations

Two gaps occurring at the same place

O. Nummi, J. Kinnunen, P. Törmä, in preparation
Phase diagram: trap frequency ratio

(Left) Local density approximation (LDA) [1] (Right) BdG results. Narrow superfluid regions do not exist in BdG unlike in LDA. BdG predicts clear coexistence, LDA not.

Phase diagram: polarization

Note the FFLO-oscillations (the deep blue)
Phase diagram: temperature

Disappearance of $\Delta_{12}$ as temperature increases and corresponding increase of $\Delta_{23}$
Three component BdG study: summary

• Narrow spatial regions of superfluids in the trap, predicted by LDA, disappear in the BdG treatment

• BdG predicts coexistence of two superfluids over large spatial regions and parameter ranges, which LDA never does. Reasons? Finite size effects? The trap stabilizes the superfluids? Others???

• Three-component mixtures non-trivial even at the mean field level

• These are preliminary, unpublished results
• Three-component mixtures, attractive interactions: a BdG study in a trap (preliminary)
• **The repulsive Hubbard model in a one-dimensional lattice: exact hopping modulation dynamics**
• RF-spectroscopy in a 1D system: signatures of the FFLO state
• Fermi condesates as sensors
Motivation:
● Recent modulation experiments in ETH [R. Jördens, N. Strohmaier, K. Gunter, H. Moritz, T. Esslinger, Nature 2008; 2009 arXiv by Esslinger and Demler groups], and description of high $T_c$ superconductors.

What we have done:
● Exact numerical simulation (TEBD code) of the modulation of a 1D Hubbard chain both in presence of harmonic confinement and for open boundary conditions;
● interpretation of the results in terms of Bethe ansatz equations;
● connection between double occupancy spectrum and energy spectrum.


Hubbard Hamiltonian with parabolic confining potential:

\[ H = -J \sum_{i,\sigma} c_{i,\sigma}^\dagger c_{i+1,\sigma} + h.c. \]

\[ + U \sum_i n_i \uparrow n_i \downarrow + \sum_i V_i (n_i \uparrow + n_i \downarrow) \]

Hopping modulation:

\[ J = J_0 + \delta J \sin(\omega t) \]
Monitoring double occupancy as a function of time

\[ t < \frac{2\pi}{\Delta E} \]

\[ \sum_i \langle n_i^+ n_i^+ \rangle \]

\[ t > \frac{2\pi}{\Delta E} \]
\[ t < \frac{2\pi}{\Delta E} \] Upper Hubbard band perceived as a continuum

\[ P(t) \propto \delta J \frac{\sin^2 [(U - \hbar \omega) t / 2\hbar]}{(U - \hbar \omega)^2} \]

\[ t > \frac{2\pi}{\Delta E} \] Detailed structure can be resolved

\[ \sum_n P_n(t) = \left( \frac{2\pi}{\hbar} \right) |V_{n,gs}|^2 \rho(E_n) t \bigg|_{E_n \simeq E_{gs} \pm \hbar \omega} \]
Simplified model ->
• no parabolic potential
• limit $U/J = \infty$

Solution ->
Bethe ansatz equations for open boundary conditions:

\[ 2Lk_j = 2\pi I_j - 2k_j - \sum_{\beta=1}^{M} \left[ \Phi \left( 2 \frac{\sin(k_j) - \lambda_{\beta}}{u} \right) + \Phi \left( 2 \frac{\sin(k_j) + \lambda_{\beta}}{u} \right) \right] \]

\[ \sum_{j=1}^{N} \left[ \Phi \left( 2 \frac{\lambda_{\alpha} - \sin(k_j)}{u} \right) + \Phi \left( 2 \frac{\lambda_{\alpha} + \sin(k_j)}{u} \right) \right] = \]

\[ 2\pi J_{\alpha} + \sum_{\beta=1(\beta \neq \alpha)}^{M} \left[ \Phi \left( \frac{\lambda_{\alpha} - \lambda_{\beta}}{u} \right) + \Phi \left( \frac{\lambda_{\alpha} + \lambda_{\beta}}{u} \right) \right] \]

where

\[ j = 1, \ldots, N, \ \alpha = 1, \ldots, M, \ I_j, \ J_{\alpha} \in \mathbb{N} \]

\[ \Phi(x) = 2 \tan^{-1}(2x) \]
$U/J \to \infty$

$$k_j = \frac{\pi}{L+1} j$$

$$\Delta E = -2J(\cos(k_p) - \cos(k_h)) + U$$

Due to selection rules many excitation energies do not correspond to any peak in the d.o. spectrum
The same approach for the confined system: According to [A.M. Rey, G. Pupillo, C.W. Clark, C.J. Williams, PRA 2005], the spectrum of a spinless fermion in presence of lattice + parabolic confinement is given by:

\[ E_i - E_0 = 2\sqrt{J\Omega}(i + 1/2) - \frac{\Omega}{32} \left[ (2i + 1)^2 + 1 - \frac{(2i + 1)^3 + 3(2i + 1)}{32\sqrt{J/\Omega}} \right] \quad i < i_c, \]

\[ E_{i=2r} \approx E_{i=2r-1} \approx \Omega r^2 + \frac{2J}{(2r)^2 - 1} \quad i > i_c. \]

\[ \frac{U}{J} = 60 \]

\[ \frac{U}{J} = 20 \]
Hopping modulation simulations in 1D repulsive Hubbard model

- Fine features in the spectrum appear after long enough modulation times
- The peak structure explained using Bethe ansatz; selection rules present
- Discreteness of the spectrum here due to 1) Finite system 2) Trap potential
- Implications to 3D: the AFM gap could be observed by modulation experiments after long enough modulation times (for present values about 1 s, but would be shorter for smaller $U/J^2$)

• Three-component mixtures, attractive interactions: a BdG study in a trap (preliminary)
• The repulsive Hubbard model in a one-dimensional lattice: exact hopping modulation dynamics
• RF-spectroscopy in a 1D system: signatures of the FFLO state
• Fermi condesates as sensors
Imbalanced/Polarized Fermi gases

Pairing between particles with unequal mass or unequal total number

Related to, e.g., high energy physics (colour superconductivity of quarks)


Experiments:


Polarization

\[ P = \frac{N_{\uparrow} - N_{\downarrow}}{N_{\uparrow} + N_{\downarrow}} \]
COULD ONE OBSERVE THE FFLO STATE IN ULTRACOLD GASES?

- FFLO (Fulde, Ferrel, Larkin, Ovchinnikov) state
  - Finite polarization $P$ and superfluidity simultaneously (also at $T=0$)
  - Non-uniform order parameter

- Observations under debate

- The parameter window for existence of this phase is exceedingly small for particles in free space, in 3D
  - Exceedingly small for particles in free space, in 3D. See e.g. D.L. Sheehy, L. Radzihovsky, PRL 2006
  - … and especially in 1D
Mean-field study of 1D Fermi gas: BdG formalism

The 1D (attractive) Fermi gas in optical lattices can be studied at mean-field level by the following (Hubbard) model:

\[
H_{mf} = -t \sum_{i,\sigma} \left( \hat{c}^\dagger_{i\sigma} \hat{c}^\dagger_{i+1\sigma} + h.c \right) + \sum_i \left( \Delta_i \hat{c}^\dagger_{i\uparrow} \hat{c}^\dagger_{i\downarrow} + h.c \right) + \sum_{i\sigma} \left( V^\text{ext}_i - \mu_\sigma \right) \hat{n}_{i\sigma}
\]

where the pairing is defined as

\[
\Delta_i = -U < \hat{c}^\dagger_{i\downarrow} \hat{c}^\dagger_{i\uparrow} >
\]

The Hamiltonian can be diagonalized using Bogoliubov transformation

\[
\hat{c}^\dagger_{i\sigma} = \sum_\alpha \left( u_{\alpha i\sigma} \hat{\gamma}_{\alpha\sigma} - \sigma v^*_{\alpha i\sigma} \hat{\gamma}^\dagger_{\alpha\bar{\sigma}} \right)
\]

The Hamiltonian then can be expressed in the following compact form

\[
\sum_{j=1}^N \begin{pmatrix} H^\sigma_{ij} & \Delta_{ij} \\ \Delta^*_{ij} & -H^\sigma_{ij} \end{pmatrix} \begin{pmatrix} u_{\alpha j\sigma} \\ v_{\alpha j\bar{\sigma}} \end{pmatrix} = E_{\alpha\sigma} \begin{pmatrix} u_{\alpha i\sigma} \\ v_{\alpha i\bar{\sigma}} \end{pmatrix}
\]

\[
H^\sigma_{ij} = -t \delta_{i,i\pm1} + (V^\text{ext}_i - \mu_\sigma) \delta_{ij}
\]

M.R. Bakhtiari, M.J. Leskinen, P. Törmä, PRL 2008
Densities and order parameter for an imbalanced gas in 1D lattice combined with external harmonic trapping

\[ P = 0.23 \]
$P = 0.7$

The diagram shows the distribution of fermionic atoms in a lattice, with different lines representing different parameters: $n_{\uparrow}^\uparrow$, $n_{\downarrow}^\downarrow$, $\Delta$, and $3 \times u_{\alpha}^2_{\uparrow}$. The y-axis represents the values of these parameters, and the x-axis represents the lattice site $i$, with the trap center indicated at the bottom.
RF-spectroscopy

| 1 > | 2 > (and |f>)

no interactions

| 1 >, | 2 > (and |f>)

interacting

\[ N_F \]

\[ \delta \]
The Hamiltonian

\[ H = \sum_{\sigma=1,2,f} \int dr \, \Psi_\sigma^\dagger(r) \left[ -\frac{\nabla^2}{2m} - \mu_\sigma \right] \Psi_\sigma(r) + U_{12} \int dr \, \Psi_1^\dagger(r) \Psi_2^\dagger(r) \Psi_2(r) \Psi_1(r) + U_{1f} \int dr \, \Psi_1^\dagger(r) \Psi_f^\dagger(r) \Psi_f(r) \Psi_1(r) + \Omega \int dr \, \Psi_f^\dagger(r) \Psi_2(r) + \Omega \int dr \, \Psi_2^\dagger(r) \Psi_f(r) + \frac{\delta}{2} \int dr \, [\Psi_2^\dagger(r) \Psi_2(r) - \Psi_f^\dagger(r) \Psi_f(r)] \]

Linear response

\[ \chi''(\delta) = \Im \left[ -i \int dr \langle T[\psi_2^\dagger(r, t) \psi_f(r, t) \psi_f^\dagger(0, 0) \psi_2(0, 0)] \rangle \right] \]

Fermi Golden rule

\[ \chi''(\delta) \propto \langle \Psi_2^\dagger \Psi_2 \rangle \langle \Psi_3^\dagger \Psi_3 \rangle = G_2 G_3 \]

\[ \delta_{th} \simeq (U_{1f} - U_{12}) n_1 + \frac{\Delta^2}{2E_F} \]
Spectral signatures of the FFLO state in 1D optical lattices

M.R. Bakhtiari, M.J. Leskinen, P. Törmä, PRL 2008
\[ J_{1/4}(\delta, K) = -2\pi \sum_{\alpha=1}^{L} \left[ \left| \sum_{i=1}^{L} v_{\alpha i/\uparrow} v_{K i\uparrow}^{\text{non}} \right|^2 n_F(E_{\alpha i/\uparrow}) \right] \delta(E_{\alpha i/\uparrow} + \epsilon_K - \delta - \mu_{1/4}) + \left| \sum_{i=1}^{L} u_{\alpha i/\uparrow} v_{K i\uparrow}^{\text{non}} \right|^2 n_F(E_{\alpha i/\downarrow}) \delta(E_{\alpha i/\downarrow} - \epsilon_K + \delta + \mu_{1/4}) \].

Andreev states at the nodes of the order parameter

Momentum conservation in the spectroscopy

Spectra at negative detunings

(1)
Signatures at negative detunings are related to strongly oscillating order parameter
Spectral weight at the negative detunings is a direct signature of Andreev bound states and of the FFLO state.
Exact numerical studies (TEBD) of the ground state and the RF spectroscopy dynamics (spectra)

- Density profiles
  \[ N_1 = 4 \quad N_2 = 20 \]
  \[ U = -20 \text{ J} \]
  \[ V_{\text{ext}} \approx 0.00016 \]

- Spectrum
  \[ \Omega = 0.1 \quad T = 10 \]
  \[ \chi = 80 \text{ (Schmidt number)} \]

---

c.f. R.A. Molina, J. Dukelsky, P. Schmitteckert, Comment PRL 2009
• Spectrum: $U = -8 \ J \quad \Omega = 0.1 \quad T = 5$ 
  $\chi = 80$ (Schmidt number)
1D FFLO signatures in RF spectroscopy

• Spectral weight at the negative detunings is a direct signature of Andreev bound states and of the FFLO state

• Predicted both by mean-field and exact numerical studies
• Three-component mixtures, attractive interactions: a BdG study in a trap (preliminary)
• The repulsive Hubbard model in a one-dimensional lattice: exact hopping modulation dynamics
• RF-spectroscopy in a 1D system: signatures of the FFLO state
• Fermi condesates as sensors
Fermi condensates for dynamic imaging of electromagnetic fields
T.K. Koponen, J. Pasanen, P. Törmä, PRL 2009
Summary

- Preliminary studies of three-component systems in traps: LDA and BdG produce differing results
- Exact hopping modulation dynamics in 1D lattices: two timescales; the structured double occupation spectrum accurately reveals the discrete energies of the ground state; implications for the observation of the AFM gap
- Observation of FFLO and Andreev bound states by RF-spectroscopy
- Fermi condesates as sensors: gap provides frequency selection